

# Stabbing circles for sets of segments in the plane<sup>\*</sup>

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**Abstract.** Stabbing a set  $S$  of  $n$  segments in the plane by a line is a well-known problem. In this paper we consider the variation where the stabbing object is a circle instead of a line. We show that the problem is tightly connected to two cluster Voronoi diagrams, in particular, the Hausdorff and the farthest-color Voronoi diagram. Based on these diagrams, we provide a method to compute a representation of all the combinatorially different stabbing circles for  $S$ , and the stabbing circles with maximum and minimum radius. We give conditions under which our method is fast. These conditions are satisfied if the segments in  $S$  are parallel, resulting in a  $O(n \log^2 n)$  time and  $O(n)$  space algorithm. We also observe that the stabbing circle problem for  $S$  can be solved in worst-case optimal  $O(n^2)$  time and space by reducing the problem to computing the stabbing planes for a set of segments in 3D. Finally we show that the problem of computing the stabbing circle of minimum radius for a set of  $n$  parallel segments of equal length has an  $\Omega(n \log n)$  lower bound.

## 1 Introduction

Let  $S$  be a set of  $n$  line segments (segments for short) in the plane. We say that a region  $\mathcal{R} \subseteq \mathbb{R}^2$  is a *stabbing region* for  $S$  if *exactly* one endpoint of each segment of  $S$  lies in the exterior of  $\mathcal{R}$ . The boundary of  $\mathcal{R}$  (also known as a *stabber* for  $S$ ) intersects all the segments in  $S$  and separates/classifies their endpoints into two classes, depending on whether or not they lie in the exterior of  $\mathcal{R}$ . Two stabbing regions  $\mathcal{R}_1$  and  $\mathcal{R}_2$  for  $S$  are *combinatorially different* if they classify the endpoints of  $S$  differently.

A natural problem is to determine the existence and compute (when possible) a representation of all combinatorially different stabbing regions for  $S$ . We are interested in stabbing regions whose boundary has constant complexity. Perhaps the simplest such region is a halfplane bounded by a line that intersects or *stabs* all the segments. Edelsbrunner et al. [15] presented an optimal  $\Theta(n \log n)$  time algorithm to compute a representation of all the  $O(n)$  combinatorially different stabbing lines for  $S$ . An  $\Omega(n \log n)$  lower bound for the decision problem was later presented by Avis et al. [6]. For parallel segments the problem can be solved in  $O(n)$  time by linear programming. If no stabbing halfplane for  $S$  exists, it is natural to ask for other types of stabbers. Computing

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all the combinatorially different stabbing wedges (regions defined by the intersection of two halfplanes) can be carried out in  $O(n^3 \log n)$  time and  $O(n^2)$  space [11]. The same question can be answered in  $O(n \log n)$  time and  $O(n)$  space for isothetic stabbing strips, quadrants and 3-sided rectangles; and in  $O(n^2 \log n)$  time and  $O(n^2)$  space for isothetic stabbing rectangles [12].

In this paper, we focus on the *stabbing circle problem*, formulated as follows. Let  $S$  be a set of  $n$  segments in the plane in general position (segments have  $2n$  distinct endpoints, no three endpoints are collinear, and no four of them are cocircular). A circle  $c$  is a stabbing circle for  $S$  if exactly one endpoint of each segment in  $S$  is contained in the exterior of the closed disk (region) induced by  $c$ ; see Figure 1. The stabbing circle problem for  $S$  consists of (1) answering whether a stabbing circle for  $S$  exists; (2) reporting a representation (for the centers) of all the combinatorially different stabbing circles for  $S$ ; and (3) finding stabbing circles with minimum and maximum radius. Note that stabbing circles of minimum radius do not always exist. On the contrary, there are cases in which any stabbing circle can be shrunk by decreasing its radius or moving its center, however, the “limit” circle is not stabbing anymore. In such cases, our task is to find this “limit” circle. The same may happen with the stabbing circles of maximum radius; refer to Lemma 21 for the details. We remark that this can happen even if the general position assumptions used in this paper are fulfilled. Note also that our stabbing criterion uses only the segment endpoints, thus,  $S$  can be seen as a set of pairs of points, where a segment is simply a convenient representation for such a pair.



**Fig. 1.** Left: Segment set with a stabbing circle. Right: Segment set with no stabbing circle.

Other works with similar criteria are as follows: Rappaport [25] considers the problem of computing the stabbing simple polygon of minimum perimeter for a set  $S$  of general segments, where a simple polygon stabs  $S$  if *at least* one point (which is not necessarily an endpoint) of each segment is in the polygon; this minimum stabbing polygon is always a convex polygon. Díaz-Báñez et al. [14] focus on computing the stabbing simple polygons of minimum perimeter or area with a distinct criterion, specifically, that at least one endpoint of each segment is required to be in the polygon. Keeping the same criterion and replacing simple polygons by disks leads to the so-called *pairs of points  $L_2$  1-center problem*, studied by Arkin et al. [3]; the goal of the problem is to determine a minimum-radius disk that contains at least one endpoint of each segment. Finally, Arkin et al. [2] consider, given a collection of compact sets, whether there exists a convex body  $\mathcal{R}$  whose boundary intersects every set in the collection. They show that, for segment sets, deciding the existence of a convex stabber is NP-hard.

72 *Our results.* First, we point out a connection between the stabbing circle problem and  
73 two *cluster Voronoi diagrams*: the *Hausdorff* and the *farthest-color Voronoi diagram*.  
74 This connection is interesting in its own right and it forms the base of our method to  
75 solve the stabbing circle problem. For a family of clusters (sets) of points, the Hausdorff  
76 Voronoi diagram (HVD) is a subdivision of the plane into regions such that every point  
77 within one region has the same nearest cluster, where the distance between a point  
78  $p \in \mathbb{R}^2$  and a cluster  $C$  is the maximum distance between  $p$  and all points in  $C$ . The  
79 farthest-color Voronoi diagram (FCVD) is the reverse: it reveals the farthest cluster  
80 for every point in a region, according to the minimum distance between a point and a  
81 cluster. Both diagrams have quadratic structural complexity in the worst case [1,16,19].  
82 However, for some classes of input sites the diagrams are linear and can be constructed  
83 efficiently, see e.g. [8,23] for the HVD. Here, clusters are the pairs of segment endpoints,  
84 and  $S$  is a family of such pairs of points.

85 Our central object is  $\text{FCVD}^*(S)$ , defined as the locus of points whose farthest-color  
86 neighbor (i.e., their *owner* in the farthest-color Voronoi diagram) is closer than their  
87 nearest cluster (i.e., their *owner* in the Hausdorff Voronoi diagram). We observe that  
88 any point  $p \in \mathbb{R}^2$  that is the center of a stabbing circle for  $S$  lies in the interior of  
89  $\text{FCVD}^*(S)$ . The points in the interior of  $\text{FCVD}^*(S)$  that are not centers of stabbing  
90 circles for  $S$  are separators among centers of combinatorially different stabbing circles.

91 Thus,  $\text{FCVD}^*(S)$  provides all the information that is relevant to stabbing circles:  
92 whether such circles exist, a list of all combinatorially different stabbing circles, and the  
93 stabbing circles with minimum and maximum radius. We identify sufficient conditions  
94 for efficient algorithms to construct  $\text{FCVD}^*(S)$ , and thus, to solve the stabbing circle  
95 problem. These conditions are: (1) the Hausdorff Voronoi diagram and the farthest-  
96 color Voronoi diagram have linear structural complexity and can be constructed fast;  
97 (2) the edges of the Hausdorff Voronoi diagram are not “spoiled” many times, where by  
98 “spoiling” an edge  $e$  we mean a technical condition necessary to cause  $e \cap \text{FCVD}^*(S)$   
99 to be disconnected. If the segments in  $S$  are parallel, conditions (1) and (2) are satisfied,  
100 and we obtain that the stabbing circle problem for  $S$  can be solved in  $O(n \log^2 n)$  time  
101 and  $O(n)$  space. As a byproduct, we establish that the farthest-color Voronoi diagram  
102 for such a set  $S$  has structural complexity  $O(n)$  and can be constructed in  $O(n \log n)$   
103 time and  $O(n)$  space, which was not previously known. In addition, we show that the  
104 problem of computing the stabbing circle of minimum radius for a set of  $n$  parallel  
105 segments of equal length has an  $\Omega(n \log n)$  lower bound.

106 *Summary.* In Section 2 we give the necessary definitions; in addition, we observe  
107 that, using a known technique, the stabbing circle problem for arbitrary segments  
108 can be solved in  $O(n^2)$  time and space. In Section 3 we show the connection of  
109  $\text{FCVD}^*(S)$  with the problem, and we give useful properties of  $\text{HVD}(S)$ ,  $\text{FCVD}(S)$ ,  
110 and  $\text{FCVD}^*(S)$ . In Section 4 we present an algorithm to compute  $\text{FCVD}^*(S)$ . In Sec-  
111 tion 5, we show that the stabbing circle problem for parallel segments can be solved  
112 in  $O(n \log^2 n)$  time and  $O(n)$  space. A lower bound for the problem of computing a  
113 stabbing circle of minimum radius for parallel segments of equal length is shown in  
114 Section 6. Finally, in Section 7, we summarize and propose questions for future work.

## 2 Preliminaries and Definitions

In what follows,  $xx'$  denotes either a pair of points or a segment as convenient. For a pair  $x, y$  of points in the plane, let  $d(x, y)$  denote the Euclidean distance between them, and let  $bis(x, y)$  denote the perpendicular bisector of the segment  $xy$ . For a region  $f \subset \mathbb{R}^2$ , we denote its boundary as  $\partial f$ , and its closure as  $\bar{f}$ .

**Definition 1.** [16,23] *The Hausdorff Voronoi diagram of  $S$  is a partitioning of  $\mathbb{R}^2$  into regions defined as follows:*

$$\begin{aligned} \text{hreg}(aa') &= \{p \in \mathbb{R}^2 \mid \forall bb' \in S \setminus \{aa'\}: \max\{d(p, a), d(p, a')\} < \max\{d(p, b), d(p, b')\}\}; \\ \text{hreg}(a) &= \{p \in \text{hreg}(aa') \mid d(p, a) > d(p, a')\}. \end{aligned}$$

Note that  $\text{hreg}(a)$  and  $\text{hreg}(a')$  are subregions of  $\text{hreg}(aa')$  (see Figure 2a). Note as well, that  $\text{hreg}(aa')$  may have several connected components. Let  $\text{HVD}(S)$  denote the graph structure of the Hausdorff Voronoi diagram of  $S$ :

$$\text{HVD}(S) = \mathbb{R}^2 \setminus \bigcup_{aa' \in S} (\text{hreg}(a) \cup \text{hreg}(a')).$$

An edge of  $\text{HVD}(S)$  is called *pure* if it separates the regions of two distinct segments, and it is called *internal* if it separates the subregions of the same segment. A vertex of  $\text{HVD}(S)$  is called *pure* if it is incident to three pure edges, and it is called *mixed* if it is incident to an internal edge. The pure vertices are defined by three distinct sites, and the mixed vertices by two distinct sites.<sup>4</sup>

**Definition 2.** [1,19] *The farthest-color Voronoi diagram is a partitioning of  $\mathbb{R}^2$  into regions defined as follows:*

$$\begin{aligned} \text{fcreg}(aa') &= \{p \in \mathbb{R}^2 \mid \forall bb' \in S \setminus \{aa'\}: \min\{d(p, a), d(p, a')\} > \min\{d(p, b), d(p, b')\}\}; \\ \text{fcreg}(a) &= \{p \in \text{fcreg}(aa') \mid d(p, a) < d(p, a')\}. \end{aligned}$$

The graph structure of this diagram is denoted as  $\text{FCVD}(S)$ :

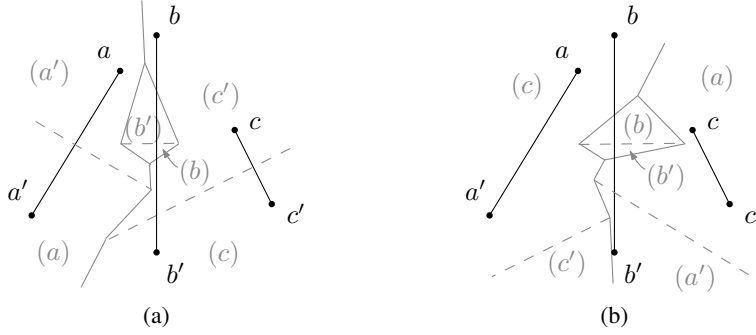
$$\text{FCVD}(S) = \mathbb{R}^2 \setminus \bigcup_{aa' \in S} (\text{fcreg}(a) \cup \text{fcreg}(a')).$$

Similarly to the case of  $\text{HVD}(S)$ ,  $\text{fcreg}(a)$  and  $\text{fcreg}(a')$  are subregions of  $\text{fcreg}(aa')$ , and  $\text{fcreg}(aa')$  may have several components. The edges and vertices of  $\text{FCVD}(S)$  are characterized as *pure* or *internal*, and *pure* or *mixed*, analogously to those of  $\text{HVD}(S)$  (see Figure 2b).

Let  $\overline{\text{hreg}}(\cdot)$  and  $\overline{\text{fcreg}}(\cdot)$  denote the closures of the respective regions.

When the segments in  $S$  are pairwise disjoint, the structural complexity of  $\text{HVD}(S)$  is  $O(n)$  [16]. This particular case of pairwise disjoint segments has not been studied for  $\text{FCVD}(S)$ . For arbitrary segments, the complexity of both diagrams is  $O(n^2)$  [1,22].

<sup>4</sup> Any vertex of  $\text{HVD}(S)$  has degree three by the general position assumption that no four end-points of segments in  $S$  are cocircular.



**Fig. 2.** (a)  $\text{HVD}(S)$ , (b)  $\text{FCVD}(S)$ . Pure and internal edges are represented in solid and dashed, respectively. The gray letters in parentheses label the respective regions.

**Definition 3.** Given a point  $p$ , the Hausdorff disk of  $p$ , denoted  $D_h(p)$ , is the closed disk with center at  $p$  and radius  $d(p, a)$ , where  $p \in \overline{\text{hreg}}(a)$ . The radius of  $D_h(p)$  is called the Hausdorff radius of  $p$ , and is denoted as  $r_h(p)$ . The farthest-color disk  $D_f(p)$  and the farthest-color radius  $r_f(p)$  of  $p$  are defined analogously.

The following lemma reveals the connection between the stabbing circle problem and the two cluster Voronoi diagrams,  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ .

**Lemma 1.** Given a point  $p$ , there exists a stabbing circle centered at  $p$  if and only if  $r_f(p) < r_h(p)$ .

*Proof.* Let  $c$  be a circle centered at  $p$  with radius  $r$ , and let  $D$  be the closed disk induced by  $c$ . Recall that  $c$  is a stabbing circle if and only if (1) each segment in  $S$  has an endpoint outside  $D$ ; and (2) each segment in  $S$  has an endpoint in  $D$ . Condition (1) is equivalent to  $r < r_h(p)$ . Condition (2) is equivalent to  $r_f(p) \leq r$ . The claim follows.  $\square$

Now we are ready to define  $\text{FCVD}^*(S)$ , which is the closure of the locus of the centers of all stabbing circles for  $S$ .

**Definition 4.** The  $\text{FCVD}^*(S)$  is the locus of points in  $\mathbb{R}^2$  for which the farthest-color radius is less than or equal to the Hausdorff radius, i.e.,  $\text{FCVD}^*(S) = \{p \in \mathbb{R}^2 : r_f(p) \leq r_h(p)\}$ .

For any point on the boundary of  $\text{FCVD}^*(S)$ , its Hausdorff radius equals its farthest-color radius. Note that this equality also holds for some points in the interior of  $\text{FCVD}^*(S)$ . Any point  $p$  in the interior of  $\text{FCVD}^*(S)$  such that  $r_h(p) = r_f(p)$  lies on an internal edge of both  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ , which separates centers of stabbing circles of two combinatorially different types, refer to Section 3.2 for more details.

By applying the transformation of Edelsbrunner and Seidel [18], both  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  can be viewed as envelopes of wedges in 3D: Lift up the pairs of endpoints of the segments in  $S$  onto the unit paraboloid  $U$ , and join the lifted endpoints obtaining a set  $S'$  of segments in 3D. Each lifted endpoint  $a$  is mapped to a hyperplane tangent

157 to  $U$  at point  $a$ . Thus, a segment in  $S$  is transformed to a pair of planes in 3D. The  
 158 lower (resp., upper) envelope of such a pair forms a *lower* and (resp., *upper*) *wedge*, re-  
 159 spectively. Then  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  correspond to the upper envelope of all lower  
 160 wedges, and to the lower envelope of all upper wedges, respectively. This transforma-  
 161 tion is explicitly given for  $\text{HVD}(S)$  in Edelsbrunner et al. [16]. Thus,  $\text{FCVD}^*(S)$   
 162 corresponds to the locus of points below  $\text{HVD}(S)$  and above  $\text{FCVD}(S)$ .

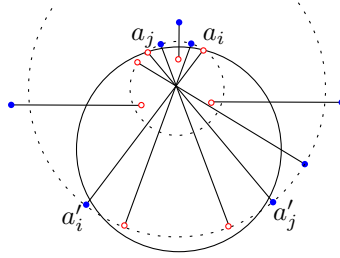
163 The locus of points between two surfaces, defined by the above construction, is  
 164 shown to be a representation of all combinatorially different stabbing planes for  $S'$  (if  
 165 one exists) [16]. Further, the authors show that this locus is a set of  $O(n^2)$  convex cells  
 166 in 3D with  $O(n^2)$  total complexity, and can be computed in  $O(n^2)$  time and space.

167 We observe that a stabbing circle for  $S$  can be transformed into a stabbing plane for  
 168  $S'$  and vice versa.

169 We obtain the following result which, to the best of our knowledge, has not been  
 170 explicitly stated anywhere before:

171 **Theorem 1.** *The stabbing circle problem for a set  $S$  of  $n$  arbitrary segments can be*  
 172 *solved in  $O(n^2)$  time and space.*

173 Claverol [10] showed that a set  $S$  of segments might have  $\Theta(n^2)$  combinatorially  
 174 different stabbing circles; see Figure 3. In the construction, each pair  $\{a_i, a_j\}$  of points  
 175 in the upper arc defines a stabbing circle that leaves the endpoints in the upper arc  
 176 between  $a_i$  and  $a_j$  outside the circle. Hence, the  $\Theta(n^2)$  stabbing circles defined in this  
 177 way are combinatorially different. We remark that it is possible to perform a small  
 178 perturbation so that the general position assumptions are satisfied.



**Fig. 3.** A set with  $\Theta(n^2)$  combinatorially different stabbing circles, and the stabbing circle defined by  $\{a_i, a_j\}$ .

### 179 3 Properties of $\text{HVD}(S)$ , $\text{FCVD}(S)$ , and $\text{FCVD}^*(S)$

180 In this section we investigate structural properties of the geometric structures involved  
 181 in the stabbing circle problem. First, in Section 3.1, we list basic properties of the Haus-  
 182 dorff and the farthest-color Voronoi diagrams. These are later used to derive structural  
 183 properties of  $\text{FCVD}^*(S)$  and the correctness of our solution to the stabbing circle prob-  
 184 lem. The structure of the Hausdorff Voronoi diagram is well known, see e.g., [16,22,23];

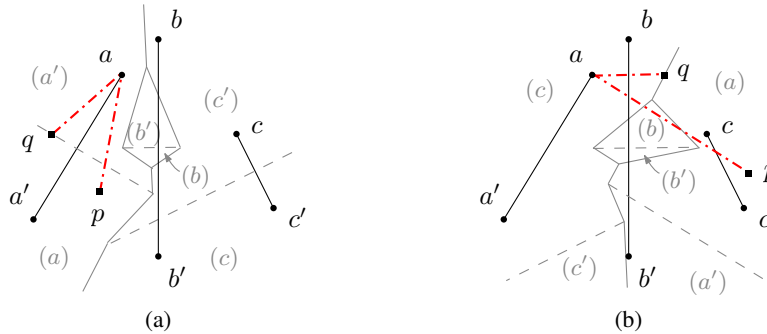
185 however, this is not the case for the farthest-color diagram. We use Section 3.1 to derive  
 186 some useful properties for  $\text{FCVD}(S)$ . Then, in Section 3.2 we investigate the structure  
 187 of  $\text{FCVD}^*(S)$ , and we characterize its faces and their complexity, linking them to fea-  
 188 tures of  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ . We show that every face of  $\text{FCVD}^*(S)$  corresponds  
 189 to a unique combinatorially distinct solution of the stabbing circle problem. Finally, in  
 190 Section 3.3, we complete the structural complexity analysis of  $\text{FCVD}^*(S)$  and count  
 191 its faces. The properties derived in this section are later used in Section 4 to obtain our  
 192 algorithm that computes all the combinatorially distinct stabbing circles for  $S$ .

### 193 3.1 Properties of $\text{HVD}(S)$ and $\text{FCVD}(S)$

194 We list some structural properties of the Hausdorff and the farthest-color Voronoi di-  
 195 agrams, which are used by our algorithms. First, a *visibility property* of both di-  
 196 agrams is summarized in the following lemma. For  $\text{HVD}(S)$ , this property follows di-  
 197 rectly from [22, Property 2] (item (a)). We prove an equivalent property for  $\text{FCVD}(S)$   
 198 (item (b)). Items (a),(b) are illustrated in Figure 4a,b respectively.

199 **Lemma 2.** Consider  $\text{hreg}(a)$  and  $\text{fcreg}(a)$ , where  $aa' \in S$  and  $|S| > 1$ .

- 200 (a) For a point  $p$  in  $\text{hreg}(a)$ , the segment  $ap$  intersects  $\partial\text{hreg}(a)$  exactly once and the  
 201 intersection point lies on an internal edge of  $\text{HVD}(S)$ . For a point  $q$  on an internal  
 202 edge of  $\partial\text{hreg}(a)$ , the segment  $aq$  does not intersect  $\text{hreg}(a)$ .  
 203 (b) For a point  $p$  in  $\text{fcreg}(a)$ , the segment  $ap$  intersects  $\partial\text{fcreg}(a)$  exactly once and the  
 204 intersection point lies on a pure edge of  $\text{FCVD}(S)$ . For a point  $q$  on a pure edge of  
 205  $\partial\text{fcreg}(a)$ , the segment  $aq$  does not intersect  $\text{fcreg}(a)$ .



**Fig. 4.** Illustration for the statement of Lemma 2 based on the diagrams from Figure 2: line segments  $ap$  and  $aq$  are shown in red dash-dotted lines.

206 *Proof.* We only prove item (b). If  $|S| > 1$ ,  $a \notin \overline{\text{fcreg}(a)}$ , thus,  $ap$  intersects  $\partial\text{fcreg}(a)$   
 207 at least once. Suppose  $ap$  intersects  $\partial\text{fcreg}(a)$  more than once, and let  $x$  and  $y$  be the  
 208 first two intersection points that are encountered when moving from  $p$  to  $a$ . Points  $x$  and

209  $y$  are on  $\partial \text{fcreg}(a)$  and the segment  $xy$  is outside  $\text{fcreg}(a)$ . Thus,  $D_f(x)$  and  $D_f(y)$   
 210 contain at least one endpoint of every segment in  $S$ , their boundary passes through  $a$ ,  
 211 and  $a'$  is outside both disks. Let  $w$  be a point on  $xy$  and let  $D(w)$  be the closed disk  
 212 centered at  $w$  whose boundary passes through  $a$ . Since the distance of any point on the  
 213 line through  $x, y$  from  $a$  increases as we move away from  $a$ ,  $D_f(y) \subset D(w) \subset D_f(x)$ .  
 214 But then  $D(w)$  must contain all points in  $D_f(y)$ ,  $a' \notin D(w)$ , and  $a \in \partial D(w)$ . Thus,  
 215  $D(w) = D_f(w)$ ; hence,  $w \in \text{fcreg}(a)$ . We obtain a contradiction.

The second claim in item (b) follows directly from the first one by considering a point  $p \in \text{fcreg}(a)$  that is infinitesimally close to  $q$ .  $\square$

216 Figure 2 illustrates a Hausdorff and a farthest-color Voronoi diagram. Every compo-  
 217 nent of a Hausdorff region  $\text{hreg}(aa')$  contains exactly one internal edge [22, Property 3]  
 218 (see the dashed lines in Figure 2a). In the following lemmas we show properties of a  
 219 similar nature for  $\text{FCVD}(S)$ .

220 **Lemma 3.** *Any bounded face of  $\text{fcreg}(aa')$  contains an internal edge. (See the faces of*  
 221 *region  $\text{fcreg}(bb')$  in Figure 2b.)*

*Proof.* Let  $f$  be a bounded face of  $\text{fcreg}(aa')$ . Suppose for the sake of contradiction that  $f$  contains no internal edge. Then  $f$  is a face of  $\text{fcreg}(a)$ , for an endpoint  $a$  of  $aa'$ , such that  $\partial f$  consists solely of pure edges. Let  $p$  be a point in  $f$  and consider the ray  $r$  from  $a$  through  $p$ . Let  $q$  be the first intersection point between  $r$  and  $\partial f$ , as we move on  $r$  starting at  $p$  away from  $a$  (such point exists because  $f$  is bounded). Since  $f$  consists solely of pure edges,  $q$  is a point on a pure edge and  $pq \in f$ , yielding a contradiction to the last claim of Lemma 2b.  $\square$

222 **Lemma 4.** *For any face of  $\text{fcreg}(a)$ , the portion of its boundary that is formed by pure*  
 223 *edges is connected. (See the solid lines in Figure 2b.)*

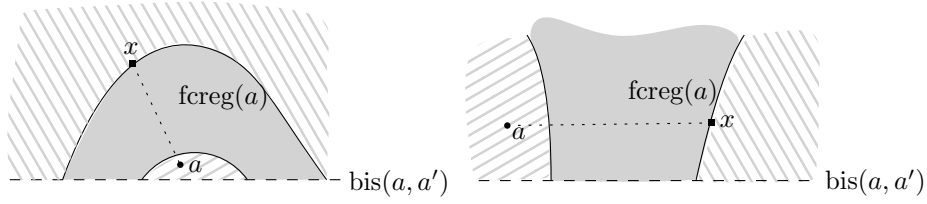
*Proof.* Suppose  $|S| > 1$  as otherwise the claim holds trivially. Let  $f$  be a face of  $\text{fcreg}(a)$ . Clearly, any internal edge on  $\partial f$  is a portion of  $\text{bis}(a, a')$ . Since  $|S| > 1$ ,  $a \notin \text{fcreg}(a)$ , but  $a$  lies in the same halfplane induced by  $\text{bis}(a, a')$  as  $\text{fcreg}(a)$ . This implies that  $a$  lies in a region of the plane bounded by  $\text{bis}(a, a')$  and one of the connected components of pure edges of  $\partial f$  (such regions are shown with tiling patterns in Figure 5). If  $\partial f$  contained more than one such connected components of pure edges, then any point  $x$  on any additional component would violate Lemma 2b. See Figure 5.  $\square$

224 The following property of  $\text{FCVD}(S)$  is only used in Section 5.

225 **Lemma 5.**  *$\text{FCVD}(S)$  has  $O(n)$  unbounded faces.*

*Proof.* We observe that each unbounded face of  $\text{FCVD}(S)$  corresponds exactly to one face of the *farthest-segment Voronoi diagram* of  $S$ ,  $\text{FsVD}(S)$ . To make the correspondence one-to-one we remove the internal unbounded edges of  $\text{FCVD}(S)$  and merge the incident faces of  $\text{fcreg}(a)$  and  $\text{fcreg}(a')$ , for every  $aa' \in S$ , into one face of  $\text{fcreg}(aa')$ . Then a face  $f$  in  $\text{FCVD}(S)$  is unbounded in a direction  $\phi$  if and only if a face  $f'$  in  $\text{FsVD}(S)$  is unbounded in the same direction  $\phi$ . See [24] for the properties of the faces





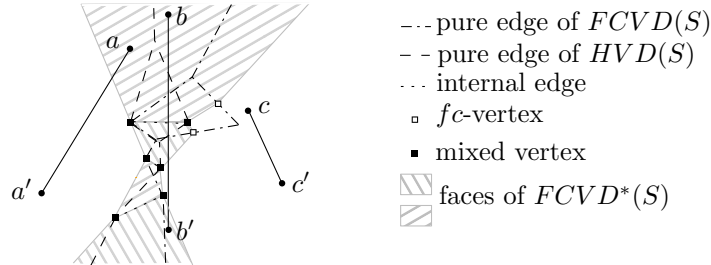
**Fig. 5.** Illustration for Lemma 4: two variants of an impossible situation.

of  $\text{FsVD}(S)$  at infinity, which are identical to those of  $\text{FCVD}(S)$ . The total number of faces in  $\text{FsVD}(S)$  is  $O(n)$  [4], thus, the same holds for the unbounded faces of  $\text{FCVD}(S)$ . Note that a component of  $\text{fcreg}(aa')$  can contain at most two unbounded portions of  $\text{bis}(a, a')$ , thus, having removed the internal unbounded edges of  $\text{FCVD}(S)$  has no effect on the derived bound.  $\square$

### 226 3.2 Properties of $\text{FCVD}^*(S)$

227 In this section, we characterize the boundary of  $\text{FCVD}^*(S)$ , its connected components  
 228 (for brevity, *components*), and its *faces*. We observe that the faces of  $\text{FCVD}^*(S)$  are  
 229 in one-to-one correspondence with the combinatorially different solutions for the stab-  
 230 bing circle problem for  $S$ . We characterize the faces of  $\text{FCVD}^*(S)$  according to their  
 231 intersection with  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ , and this characterization forms the basis of  
 232 our algorithm to compute  $\text{FCVD}^*(S)$  (and thus to solve the stabbing circle problem)  
 233 presented in Section 4.

234 We proceed with describing the boundary of the components of  $\text{FCVD}^*(S)$ . Notice  
 235 that a component is unbounded in a direction  $\phi$  if and only if there exists a stabbing line  
 236 for  $S$  that is orthogonal to  $\phi$ .



**Fig. 6.**  $\text{FCVD}^*(S)$  where  $S$  is the set of segments  $\{aa', bb', cc'\}$  from Figure 2.

237 We first observe that the vertices of  $\partial\text{FCVD}^*(S)$  are all incident to edges of  
 238  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ . Indeed, any vertex of  $\partial\text{FCVD}^*(S)$  is caused by switching the  
 239 region of either  $\text{HVD}(S)$  or of  $\text{FCVD}(S)$  (in a traversal of  $\partial\text{FCVD}^*(S)$ ), which hap-  
 240 pens exactly when  $\partial\text{FCVD}^*(S)$  meets an edge of one of the diagrams. Thus we dis-  
 241 tinguish three types of vertices of  $\partial\text{FCVD}^*(S)$ : (1) vertices incident to pure edges of

242 HVD( $S$ ), called *h-vertices*; (2) vertices incident to pure edges of FCVD( $S$ ), called  
 243 *fc-vertices*; and (3) vertices incident to internal edges of either diagram, called *mixed*  
 244 *vertices*. See Figure 6 that illustrates faces of FCVD $^*(S)$  and vertices of  $\partial$ FCVD $^*(S)$ .  
 245 Notice that *h-vertices* and *fc-vertices* belong to the interior of corresponding pure edges.  
 246 Moreover, the boundary of FCVD $^*(S)$  intersects these pure edges transversally, thus  
 247 each h- or fc- vertex is adjacent to a portion of a pure edge of, respectively, HVD( $S$ ) or  
 248 FCVD( $S$ ) within FCVD $^*(S)$ .

249 The following lemma implies that each mixed vertex of  $\partial$ FCVD $^*(S)$  is a mixed  
 250 vertex of either HVD( $S$ ) or FCVD( $S$ ), hence, the name for such vertices. Moreover,  
 251 this lemma and its corollary lead to the definition of a *face* of FCVD $^*(S)$ , which will  
 252 then be treated as an atomic piece of FCVD $^*(S)$ .

253 **Lemma 6.** *Point  $p \in \text{bis}(a, a')$  is in FCVD $^*(S)$  if and only if  $p$  lies on an internal*  
 254 *edge of both HVD( $S$ ) and FCVD( $S$ ). If  $p$  is on  $\partial$ FCVD $^*(S)$ , then  $p$  is a mixed vertex*  
 255 *of either HVD( $S$ ) or FCVD( $S$ ).*

256 *Proof.* Since  $p \in \text{bis}(a, a')$ , either both  $a, a'$  are outside  $D_h(p)$  or they lie on its bound-  
 257 ary. Symmetrically, either both  $a, a'$  are in the interior of  $D_f(p)$  or on its boundary.

258 Suppose  $p \in \text{FCVD}^*(S)$ . Then  $D_f(p) \subseteq D_h(p)$ . By the above argument,  $D_f(p) =$   
 259  $D_h(p)$  and both  $a, a'$  lie on the boundary of this disk. Therefore,  $r_h(p) = r_f(p) =$   
 260  $pa = pa'$ , and the claim follows.

261 Suppose that  $p$  lies on an internal edge of both HVD( $S$ ) and FCVD( $S$ ) that sep-  
 262 arates the respective regions of  $a$  and of  $a'$ . Since  $p$  lies on such edge of HVD( $S$ ),  
 263  $r_h(p) = pa$ , and since  $p$  lies on such edge of FCVD( $S$ ),  $r_f(p) = pa$ . Thus  $r_h(p) =$   
 264  $r_f(p)$ , hence,  $p \in \text{FCVD}^*(S)$ .

Now we prove the second part of the statement. Since  $p \in \text{bis}(a, a')$  is on  
 $\partial$ FCVD $^*(S)$ ,  $D_h(p) = D_f(p)$ , and the boundary of this disk passes through  $a, a'$ .  
 It is easy to see that this boundary also passes through an endpoint  $c$  of some segment  
 $cc' \in S$  (otherwise, the center of this disk could move in any direction while still being  
 in FCVD $^*(S)$ , contradicting the fact that  $p \in \partial$ FCVD $^*(S)$ ). If the other endpoint  $c'$  of  
 $cc'$  is inside this disk, then  $p$  is a mixed vertex of HVD( $S$ ), and if it is outside,  $p$  is a  
 mixed vertex of FCVD( $S$ ).  $\square$

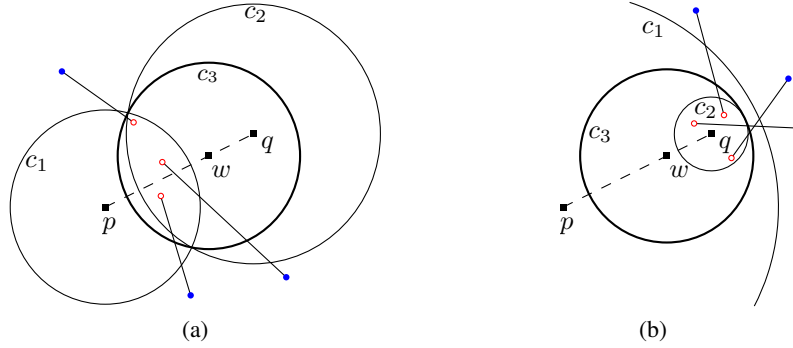
265 **Corollary 1.** *No mixed vertex of HVD( $S$ ) or FCVD( $S$ ) may lie in the interior of*  
 266 *FCVD $^*(S)$ .*

*Proof.* Suppose  $p$  is a mixed vertex of HVD( $S$ ) such that  $p \in \text{FCVD}^*(S)$ . Let  $e$  denote  
 the internal edge of HVD( $S$ ) incident to  $p$ ;  $e$  is a portion of  $\text{bis}(a, a')$ , for some  $aa' \in S$ .  
 Since  $p$  is a vertex of  $e$ , there is a point  $p'$  on  $\text{bis}(a, a')$  infinitesimally close to  $p$  such  
 that  $p'$  does not belong to  $e$ . Observe that  $r_h(p') < p'a$ , and both  $a, a'$  are outside the  
 Hausdorff disk of  $p'$ . This implies that  $p'$  is not in FCVD $^*(S)$ , and therefore  $p$  is not in  
 the interior of FCVD $^*(S)$ . A similar argument (with  $r_f(p') > p'a$ ) proves the case of  $p$   
 being a mixed vertex of FCVD( $S$ ).  $\square$

267 A component of FCVD $^*(S)$  may (or may not) contain *internal edges* that partition  
 268 the component into disjoint open *faces*. An internal edge of FCVD $^*(S)$  is the common  
 269 portion of one internal edge of HVD( $S$ ) and one such edge of FCVD( $S$ ) within a

270 component of  $\text{FCVD}^*(S)$ . A component of  $\text{FCVD}^*(S)$  that contains internal edges is  
 271 called a *multiple-face component* as it consists of more than one face. A component that  
 272 contains no internal edges is called *single-face component* and its interior is a single  
 273 face of  $\text{FCVD}^*(S)$ . To compute  $\text{FCVD}^*(S)$  we need to identify its components. In  
 274 the following we show that internal edges of  $\text{FCVD}^*(S)$  are always incident to mixed  
 275 vertices of either  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , and thus, the multiple-face components of  
 276  $\text{FCVD}^*(S)$  can be easily identified. We distinguish the single-face components in two  
 277 types: those that contain a vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ ; and those that contain no  
 278 such vertex, which are of a special form (see Lemma 11). Identifying the latter type of  
 279 single-face components poses a major difficulty to our algorithm.

280 The following lemma shows that the faces of  $\text{FCVD}^*(S)$  correspond exactly to com-  
 281 binatorially different stabbing circles. Thus, each single-face component reveals exactly  
 282 one combinatorially distinct solution for the stabbing circle problem; the multiple-face  
 283 components reveal a number of such solutions, exactly one for each of their faces.



**Fig. 7.** Illustration for the proof of Lemma 7.

284 **Lemma 7.** *Two stabbing circles are combinatorially different if and only if their centers*  
 285 *lie in different faces of  $\text{FCVD}^*(S)$ .*

286 *Proof.* Let  $c_1$  and  $c_2$  be two combinatorially different stabbing circles, and let respec-  
 287 tively  $p$  and  $q$  be their centers. There is a segment  $aa' \in S$  such that  $a$  is enclosed in  
 288  $c_1$ , and  $a'$  is enclosed in  $c_2$ . Observe that  $p$  and  $q$  lie in different halfplanes with respect  
 289 to  $\text{bis}(a, a')$ . If  $p$  and  $q$  were in the same face  $f$  of  $\text{FCVD}^*(S)$ , then there would be a  
 290 path  $\pi$  connecting  $p$  and  $q$  such that  $\pi$  lies entirely in  $f$ . Since  $p$  and  $q$  are separated by  
 291  $\text{bis}(a, a')$ , path  $\pi$  would need to cross  $\text{bis}(a, a')$  at a point  $t$ . However, point  $t$  cannot  
 292 lie inside  $f$  due to Lemma 6, which would derive a contradiction.

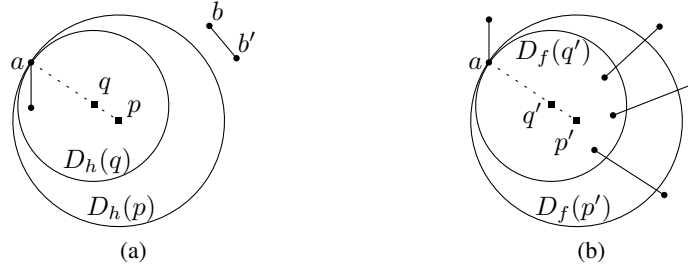
Next, suppose that the stabbing circles  $c_1$  and  $c_2$  are combinatorially equivalent. Let  
 $w$  be a point on the segment  $pq$ . We show that  $w$  is the center of a stabbing circle  $c_3$  that  
 is combinatorially equivalent to  $c_1$  and  $c_2$ . Indeed, if  $c_1$  and  $c_2$  intersect in two points,  
 then let  $c_3$  be the circle centered at  $w$  and passing through these two intersection points;  
 see Figure 7a. If  $c_2$  is enclosed in  $c_1$ , let  $c_3$  be the minimum circle centered at  $w$  and

enclosing  $c_2$ ; see Figure 7b. Let  $D_1, D_2$  and  $D_3$  be the disks corresponding to  $c_1, c_2$  and  $c_3$  respectively. Observe that  $D_1 \cap D_2 \subset D_3$  and  $((\mathbb{R}^2 \setminus D_1) \cap (\mathbb{R}^2 \setminus D_2)) \subset (\mathbb{R}^2 \setminus D_3)$ . Thus all the endpoints of segments in  $S$  that are enclosed in  $c_1$  and  $c_2$  are also enclosed in  $c_3$ , and the ones that lie outside of  $c_1$  and  $c_2$  also lie outside of  $c_3$ . This implies that  $c_3$  is a stabbing circle that is combinatorially the same as  $c_1$  and  $c_2$ . By Lemma 1,  $r_f(w) < r_h(w)$ , and thus  $w$  is in the interior of  $\text{FCVD}^*(S)$  and does not lie on an internal edge of it. This implies that the closed segment  $pq$  lies in one face of  $\text{FCVD}^*(S)$ .  $\square$

The second part of the above proof implies that, for any pair of points  $p, q$  within one face  $f$  of  $\text{FCVD}^*(S)$ , the segment  $pq$  lies in  $f$ . This proves the following property:

**Corollary 2.** *The faces of  $\text{FCVD}^*(S)$  are convex.*

$\text{FCVD}^*(S)$  has the following visibility property, which is used in the proofs of subsequent lemmas in this section.



**Fig. 8.** Illustration for the proof of Lemma 8.

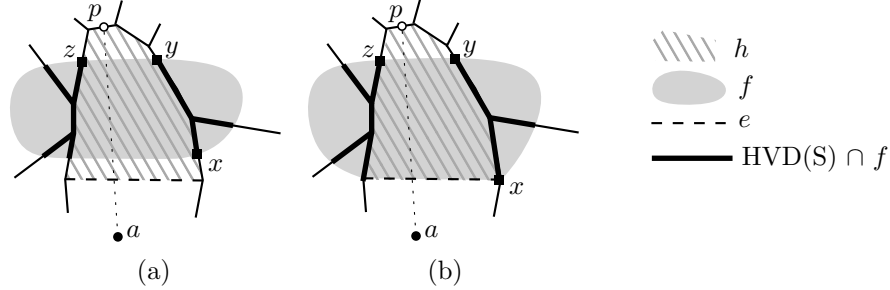
**Lemma 8.** (a) Let  $p$  be a point outside  $\text{FCVD}^*(S)$ , and let  $aa'$  be a segment in  $S$  such that  $p \in \overline{\text{hreg}}(a)$ . Then the entire segment  $(pa \cap \overline{\text{hreg}}(a))$  is outside  $\text{FCVD}^*(S)$ . (b) Let  $p'$  be a point in  $\text{FCVD}^*(S)$ , and let  $aa'$  be a segment in  $S$  such that  $p' \in \overline{\text{freg}}(a)$ . Then the entire segment  $(p'a \cap \overline{\text{freg}}(a))$  is in  $\text{FCVD}^*(S)$ .

*Proof.* By Lemma 2,  $(pa \cap \overline{\text{hreg}}(a))$  and  $(p'a \cap \overline{\text{freg}}(a))$  are segments.

(a) Let  $q$  be a point on  $pa \cap \overline{\text{hreg}}(a)$ ; see Figure 8a. Clearly,  $D_h(q) \subseteq D_h(p)$ . Since  $p \notin \text{FCVD}^*(S)$ , it follows that  $r_f(p) > r_h(p)$  and, in consequence, there exists a segment  $bb' \in S$  such that  $b, b' \notin D_h(p)$ . Since  $D_h(q) \subseteq D_h(p)$ , points  $b$  and  $b'$  are also outside  $D_h(q)$ . Thus,  $q \notin \text{FCVD}^*(S)$ .

(b) Let  $q'$  be a point in  $p'a \cap \overline{\text{freg}}(a)$ ; see Figure 8b. Since  $p' \in \text{FCVD}^*(S)$ , we have that  $r_f(p') \leq r_h(p')$  and thus each segment from  $S$  has at least one endpoint outside  $D_f(p')$  or on its boundary. Since  $D_f(q') \subset D_f(p')$ , the same holds for  $D_f(q')$ . Thus,  $D_f(q') \subset D_h(q')$ , that is,  $q' \in \text{FCVD}^*(S)$ .  $\square$

The following two lemmas analyze the intersection of a face of  $\text{FCVD}^*(S)$  with  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ . In particular, we characterize connectedness and non-emptiness of such intersections. These are properties that enable us to efficiently identify the faces of  $\text{FCVD}^*(S)$ .



**Fig. 9.** A face  $f$  of  $\text{FCVD}^*(S)$  in an impossible situation where  $f \cap \text{HVD}(S)$  is disconnected. Two variants with respect to the internal edge  $e$ : (a)  $e$  is outside  $\bar{f}$ ; (b)  $e$  is on  $\partial f$ .

**Lemma 9.** For a face  $f$  of  $\text{FCVD}^*(S)$ , both  $f \cap \text{HVD}(S)$  and  $f \cap \text{FCVD}(S)$  are connected.

*Proof.* Suppose for the sake of contradiction that  $f \cap \text{HVD}(S)$  is disconnected, see Figure 9. Then there is a face  $h$  of  $\text{hreg}(a)$  such that  $f \cap \partial h$  has at least two connected components. By [22, Property 3],  $\partial h$  contains exactly one internal edge  $e$ ,  $e \subseteq \text{bis}(a, a')$ . By the definition of a face of  $\text{FCVD}^*(S)$ ,  $e$  cannot intersect the interior of  $f$ ; it can only border  $f$  or it must be outside  $\bar{f}$ , see Figure 9a and b.

Consider the first connected component of  $f \cap \partial h$  that follows  $e$  in a counterclockwise traversal of  $\partial h$ . Let the endpoints of this component be called  $x$  and  $y$  (in the order of the traversal), see Figure 9. Let  $z$  be the first point of  $f$  encountered as we continue traversing  $\partial h$  counterclockwise beyond  $y$ . That is,  $z$  is a point in another connected component of  $f \cap \partial h$ . Consider the portions of  $\partial f$  and  $\partial h$  respectively from point  $y$  to point  $z$ . The portion of  $\partial f$  from  $y$  to  $z$  is inside  $h$ . The portion of  $\partial h$  from  $y$  to  $z$  consists solely of pure edges. Thus a point  $p$  in this portion of  $\partial h$  infinitesimally close to  $y$  is outside  $\text{FCVD}^*(S)$ . Then by Lemma 8a  $pa \cap h$  is outside of  $\text{FCVD}^*(S)$ . By Lemma 2a, the segment  $pa$  intersects  $\partial \text{hreg}(a)$  once and the intersection point lies on  $e$ . Thus,  $(pa \cap h) \cap f$  is not empty. We obtain a contradiction.

The proof for  $f \cap \text{FCVD}(S)$  is similar. In particular, if  $f \cap \text{FCVD}(S)$  is disconnected, then  $f \cap \partial h$  is disconnected, for a face  $h$  of  $\text{fcreg}(a)$ . Similarly to the above, by Lemma 4 and Lemma 2b, there is a point  $p$  on a pure edge on  $\partial \text{fcreg}(a)$  such that  $p \notin \text{FCVD}^*(S)$ , and the supporting line of the segment  $pa$  intersects  $f$  inside  $\text{fcreg}(a)$ . Let  $p'$  be a point in this intersection. Then point  $p \in p'a$ ,  $p \in \text{fcreg}(a)$  and  $p \notin \text{FCVD}^*(S)$ , which contradicts Lemma 8b.  $\square$

**Lemma 10.** Let  $f$  be a bounded face of  $\text{FCVD}^*(S)$ . (a) At least one of  $f \cap \text{HVD}(S)$  and  $f \cap \text{FCVD}(S)$  must be non-empty. (b) If one of  $f \cap \text{HVD}(S)$  or  $f \cap \text{FCVD}(S)$  is empty, then  $f$  belongs to a component of  $\text{FCVD}^*(S)$  with multiple faces.

*Proof.* To prove item (a), recall that any vertex on the boundary of a component of  $\text{FCVD}^*(S)$  is incident to an edge of  $\text{HVD}(S)$  or of  $\text{FCVD}(S)$  within that component.



**Fig. 10.** Illustration for the proof of Lemma 10. An impossible situation where a face  $f$  of  $\text{FCVD}^*(S)$  (shaded) is such that (a)  $\text{HVD}(S) \cap f$  is empty; (b)  $\text{FCVD}(S) \cap f$  is empty.

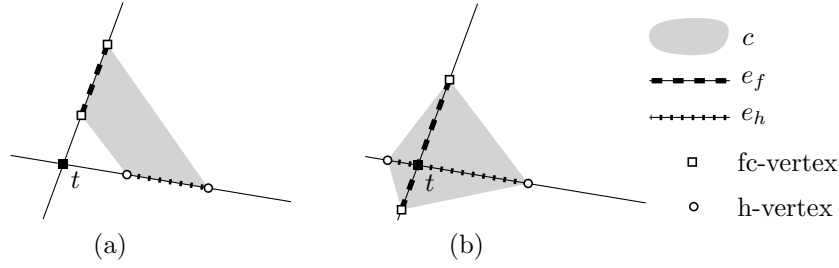
Recall also that a face of  $\text{FCVD}^*(S)$  is an open region that does not include internal edges of the diagrams, but can be incident to portions of them. Therefore any h- or fc-vertex on the boundary of a face of  $\text{FCVD}^*(S)$  is incident to a pure edge of respectively  $\text{HVD}(S)$  or  $\text{FCVD}(S)$  within this face. For the mixed vertices on the boundary of the face it might not hold, since the internal edges incident to them lie on the boundary of the face of  $\text{FCVD}^*(S)$  by definition.

Suppose for the sake of contradiction that  $f \cap \text{HVD}(S) = \emptyset$  and  $f \cap \text{FCVD}(S) = \emptyset$ . Then all the vertices of  $\text{FCVD}^*(S)$  on the boundary of  $f$  are mixed vertices. Since  $f$  is bounded and non-empty, the number of vertices on  $\partial f$  is at least four. All these mixed vertices cannot lie on the bisector of the same segment in  $S$  because  $f$  is convex (see Corollary 2). Therefore  $f$  is incident to a segment of the internal edges of at least two regions of  $\text{HVD}(S)$ . Let points  $s, t$  be respectively a point in the first segment and a point in the second one. Points  $s$  and  $t$  belong to the Hausdorff Voronoi regions of two different segments, thus the open segment  $st$  crosses at least one pure edge of  $\text{HVD}(S)$ . Since  $f$  is convex, the entire open segment  $st$  is contained in  $f$ , and thus  $f \cap \text{HVD}(S)$  is not empty. We arrive to a contradiction.

Now we prove item (b). Suppose for the sake of contradiction that  $f$  is bounded,  $f \cap \text{HVD}(S)$  is empty, and  $\bar{f}$  is a single-face component of  $\text{FCVD}^*(S)$ . Since  $f \cap \text{HVD}(S)$  is empty,  $f$  is entirely contained in  $\text{hreg}(a)$  for an endpoint  $a$  of some segment  $aa' \in S$ . Since segments in  $S$  do not share endpoints, no pure edge of  $\text{HVD}(S)$  can overlap with an edge of  $\partial \text{FCVD}^*(S)$  (they can only intersect in one point). Then there is a point  $p \in \text{hreg}(a) \setminus f$  such that  $pa$  intersects  $f$  and  $p \notin \text{FCVD}^*(S)$  (see Figure 10a). But by Lemma 8a, since  $p \notin \text{FCVD}^*(S)$  the entire segment  $(pa \cap \text{hreg}(a))$  must be outside  $\text{FCVD}^*(S)$ . We obtain a contradiction.

The symmetric statement about  $f \cap \text{FCVD}(S)$  can be shown as follows. Suppose that  $f \subset \text{fcreg}(a)$ , see Figure 10b. Similarly to the above, edges of  $\partial f$  and  $\partial \text{fcreg}(a)$  do not overlap. Pick a point  $p' \in f$ . By Lemma 8b, the entire segment  $p'a \cap \text{fcreg}(a)$  lies in  $f$ . Recall that  $a \notin \text{fcreg}(a)$  (since segments in  $S$  do not share endpoints), and thus  $f$  must intersect  $\text{FCVD}(S)$ . We obtain a contradiction.  $\square$

Finally, in the next lemma, we explore a special type of single-face components of  $\text{FCVD}^*(S)$ : the ones that contain no vertices of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ . The faces that correspond to bounded components of this type create the main difficulty in computing  $\text{FCVD}^*(S)$ .



**Fig. 11.** Illustration for the proof of Lemma 11. A bounded component  $c$  of  $\text{FCVD}^*(S)$  that does not contain a vertex of  $\text{HVD}(S)$  or of  $\text{FCVD}(S)$ : (a) an impossible situation; (b) the only possible situation.

**Lemma 11.** *Let  $c$  be a component of  $\text{FCVD}^*(S)$  that contains no vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ . If  $c$  is bounded, then  $c$  contains exactly one intersection of two pure edges (one of  $\text{HVD}(S)$  and one of  $\text{FCVD}(S)$ ), and  $\partial c$  is a quadrilateral. If  $c$  is unbounded, then  $c$  contains an unbounded portion of a pure edge of either  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ .*

*Proof.* Since  $c$  does not contain any mixed vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , by Lemma 6,  $c$  does not intersect any internal edge of the diagrams. Thus  $\partial c$  may only contain h- and fc-vertices (no mixed vertices). Further, the interior of  $c$  is a single face  $f$  of  $\text{FCVD}^*(S)$  ( $f = c$ ).

Suppose first that  $c$  is bounded. Lemmas 9 and 10 imply that  $f \cap \text{HVD}(S)$  is, respectively, connected and non-empty. Together with the fact that  $f$  contains no vertices of  $\text{HVD}(S)$ , this implies that  $f \cap \text{HVD}(S)$  is an (open) line segment, whose endpoints are two h-vertices on  $\partial c$ . By the analogous argument for  $f \cap \text{FCVD}(S)$ ,  $\partial c$  has exactly two fc-vertices. Therefore  $\partial c = \partial f$  is a quadrilateral. See Figure 11b.

Let  $e_h$  and  $e_f$  denote respectively the open line segments  $f \cap \text{HVD}(S)$  and  $f \cap \text{FCVD}(S)$ ; see Figure 11. Let  $t$  be the point of intersection between the supporting lines of  $e_h$  and  $e_f$  (the case where these lines are parallel is similar). We will show that  $t \in f$ . Note that it is impossible that an h-vertex is followed by the other h-vertex on  $\partial c$  (see Figure 11a), as it would imply that  $e_h$  and  $e_f$  lie entirely on  $\partial c$ , thus  $f \cap \text{HVD}(S)$  would be empty (as well as  $f \cap \text{FCVD}(S)$ ), what contradicts Lemma 10. Thus the different type of vertices interleave on  $\partial f$  (see Figure 11b), which implies that  $t \in f$ . This completes the proof of the first statement.

Now suppose that  $c$  is unbounded. Note that  $\partial c$  consists of at least two edges, and  $c$  is convex (see Corollary 2). Thus there are two unbounded edges on  $\partial c$  that have different supporting lines. Denote these edges as  $e$  and  $g$ . If  $c$  contained no unbounded portions of pure edges of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , then both  $e$  and  $g$  would be contained in  $\text{hreg}(a)$  and in  $\text{freg}(b)$ , for some endpoints  $a$  and  $b$  of segments in  $S$ . But then both  $e$  and  $g$  must be portions of  $\text{bis}(a, b)$ . We obtain a contradiction.  $\square$

We have explored different types of components of  $\text{FCVD}^*(S)$ , and we saw that a component may or may not be comprised of multiple faces, may or may not contain a vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , and it may or may not be bounded. The cumulative complexity of all components that contain multiple faces, or contain a vertex

of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , or are unbounded, is  $O(|\text{HVD}(S)| + |\text{FCVD}(S)|)$  (see the proof of Theorem 2 in Section 3.3). However, the bounded single-face components that do not contain a vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$  (i.e., the quadrilateral components of Lemma 11) do not fall under this bound. We bound their number in the next section. Identifying these components poses the main challenge to our algorithm in Section 4.

### 3.3 Complexity of $\text{FCVD}^*(S)$

With some abuse of notation, we denote by  $|\text{HVD}(S)|$ ,  $|\text{FCVD}(S)|$  and  $|\text{FCVD}^*(S)|$  respectively the number of edges of  $\text{HVD}(S)$ ,  $\text{FCVD}(S)$ , and  $\partial\text{FCVD}^*(S)$ . We aim to connect  $|\text{FCVD}^*(S)|$  with  $|\text{HVD}(S)|$  and  $|\text{FCVD}(S)|$ .

We classify the segments in  $S$  with respect to a portion of a fixed pure edge of  $\text{HVD}(S)$ . Let  $e$  be a connected portion of pure edge of  $\text{HVD}(S)$  that separates  $\text{hreg}(a)$  and  $\text{hreg}(b)$ , for two segments  $aa', bb' \in S$ . In particular,  $e \subseteq \text{bis}(a, b)$ . For the rest of this section, it is convenient to perform a rotation of the coordinate system so that  $e$  is horizontal. Let  $u$  and respectively  $v$  be the left and right endpoints of  $e$ .

If  $u$  is a mixed vertex of  $\text{HVD}(S)$ , we redefine  $u$  as a point on  $e$  infinitesimally to the right, so that  $u$  is in the boundary of only  $\text{hreg}(a)$  and  $\text{hreg}(b)$ . We proceed analogously with  $v$ .

The Hausdorff disks  $D_h(u)$  and  $D_h(v)$  have  $a, b$  on the boundary, they contain  $aa', bb'$ , and they do not contain any other segment of  $S$ . The same holds for the Hausdorff disk of any point of  $e$ . Hence, every segment  $cc' \in S \setminus \{aa', bb'\}$  can be classified as follows (see Figure 12, left):

- $cc'$  is of type *out* if both  $c$  and  $c'$  are outside  $D_h(u) \cup D_h(v)$ ;
- $cc'$  is of type *in* if either  $c$  or  $c'$  is contained in  $D_h(u) \cap D_h(v)$  and the other endpoint is outside  $D_h(u) \cup D_h(v)$ ;
- $cc'$  is of type *left* if either  $c$  or  $c'$  is contained in  $D_h(u) \setminus D_h(v)$  and the other endpoint is outside  $D_h(u) \cup D_h(v)$ ;
- $cc'$  is of type *right* if either  $c$  or  $c'$  is contained in  $D_h(v) \setminus D_h(u)$  and the other endpoint is outside  $D_h(u) \cup D_h(v)$ ;
- $cc'$  is of type *middle* if either  $c$  or  $c'$  is contained in  $D_h(u) \setminus D_h(v)$  and the other endpoint is contained in  $D_h(v) \setminus D_h(u)$ .

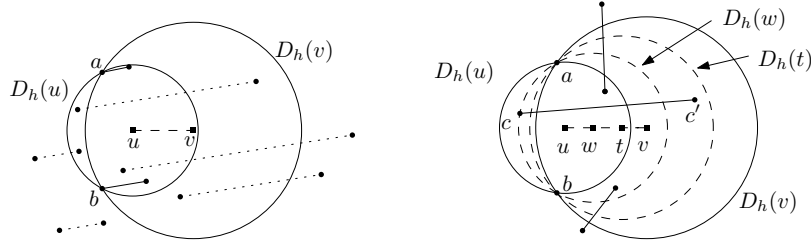
For any point  $p \in e$ , we use  $p_\ell$  and  $p_r$  to denote two points in  $e$  infinitesimally close to  $p$  and lying to the left and right of  $p$ , respectively. Additionally, let  $M_{\text{bis}}^e$  denote the set of segments  $cc' \in S$  of type *middle* for edge  $e$  such that  $e$  intersects an internal edge of  $\text{FCVD}(S)$  separating  $\text{fcreg}(c)$  from  $\text{fcreg}(c')$ . Notice that this internal edge is a portion of  $\text{bis}(c, c')$ . We also set  $m_{\text{bis}}^e = |M_{\text{bis}}^e|$ .

Let  $m_{\text{bis}}$  denote the total number of pairs formed by a pure edge  $e$  of  $\text{HVD}(S)$  and a segment  $cc' \in S$  such that  $cc' \in M_{\text{bis}}^e$ .

**Lemma 12.** *Let  $e$  be a portion of a pure edge of  $\text{HVD}(S)$ . The number of faces of  $\text{FCVD}^*(S)$  intersected by  $e$  is at most  $1 + m_{\text{bis}}^e$ .*

*Proof.* Let  $e'$  be the left-most portion (if any) of the interior of  $e$  contained in  $\text{FCVD}^*(S)$ , and let  $w$  be its right endpoint. Since  $w \in \text{FCVD}^*(S)$ , each segment





**Fig. 12.** Left: From top to bottom, the types of the dotted segments are *middle*, *left*, *in*, *right*, and *out*. Right: Illustration for the proof of Lemma 12.

427 from  $S$  has at least one endpoint inside  $D_h(w)$ . Let  $cc'$  be a segment in  $S$  such that  
 428  $w_r \in \overline{\text{fcreg}}(c)$ . Since  $w_r \notin \text{FCVD}^*(S)$ , both  $c$  and  $c'$  are outside  $D_h(w_r)$ . This implies  
 429 that one of  $c$  or  $c'$  (say,  $c$ ) lies in the portion of the boundary of  $D_h(w)$  contained in  
 430  $D_h(u)$ , and the other endpoint lies outside  $D_h(w)$  (see Figure 12, right). Consequently,  
 431 for any point  $t$  in  $w_r v$ ,  $D_h(t)$  does not contain  $c$ . Suppose that there are more portions  
 432 of  $e$  contained in  $\text{FCVD}^*(S)$ , and let  $e''$  be the left-most one. Then, for any point  $t$  in  
 433  $e''$ ,  $D_h(t)$  contains  $c'$  but not  $c$ . In particular,  $c'$  lies in  $D_h(v) \setminus D_h(u)$  and  $cc'$  is of type  
 434 *middle* for  $e$ . Hence,  $\text{bis}(c, c')$  intersects  $e$  at a point  $q$  leaving  $e'$  to its left and  $e''$  to  
 435 its right. If  $q$  lies on an internal edge of  $\text{FCVD}(S)$  separating  $\text{fcreg}(c)$  from  $\text{fcreg}(c')$ ,  
 436 then  $cc' \in M_{\text{bis}}^e$  and we assign  $e''$  to  $cc'$ .

437 Otherwise,  $q$  lies in  $\overline{\text{fcreg}}(d)$ , for some  $dd' \in S \setminus \{aa', bb', cc'\}$ . Since  $w_r \in$   
 438  $\overline{\text{fcreg}}(c)$ , the segment  $w_r q$  crosses at least one edge of  $\text{FCVD}(S)$ . We start travers-  
 439 ing the segment  $w_r v$  starting from  $w_r$ . Suppose that we leave  $\overline{\text{fcreg}}(c)$  and we enter  
 440  $\overline{\text{fcreg}}(f)$ , for some  $ff' \in S$ . Notice that this happens before we reach  $e''$ . Since  $w \in$   
 441  $\text{FCVD}^*(S)$ ,  $w_r \notin \text{FCVD}^*(S)$  and  $f \neq c'$ , the point  $f$  does not lie in  $D_h(u) \cap D_h(v)$ ,  
 442  $D_h(v) \setminus D_h(u)$  or outside  $D_h(u) \cup D_h(v)$ . Hence, it lies in  $D_h(u) \setminus D_h(v)$ . Further-  
 443 more, since  $e'' \subseteq \text{FCVD}^*(S)$ ,  $ff'$  is of type *middle* and, for any point  $t$  in  $e''$ ,  $D_h(t)$   
 444 contains  $f'$ . Hence,  $\text{bis}(f, f')$  intersects  $e$  at a point  $q'$  leaving  $e'$  to its left and  $e''$  to its  
 445 right. If  $q'$  lies on an internal edge of  $\text{FCVD}(S)$  separating  $\text{fcreg}(f)$  from  $\text{fcreg}(f')$ ,  
 446 then  $ff' \in M_{\text{bis}}^e$  and we assign  $e''$  to  $ff'$ . Otherwise, we continue traversing  $w_r v$ . The  
 447 left endpoint of  $e''$  is in the farthest-color region of a point in  $D_h(v) \setminus D_h(u)$ . Thus,  
 448 as we traverse  $w_r v$  and simultaneously  $\text{FCVD}(S)$ , at some point we cross an edge of  
 449  $\text{FCVD}(S)$  separating the farthest-color region of a point in  $D_h(u) \setminus D_h(v)$  from the  
 450 farthest-color region of a point in  $D_h(v) \setminus D_h(u)$ . This is only possible when this edge  
 451 is an internal edge of  $\text{FCVD}(S)$  separating the regions of two endpoints of a segment  
 452 of type *middle*. We assign  $ee'$  to this segment in  $M_{\text{bis}}^e$ .

Next, we select the right endpoint of  $e''$  and perform the same analysis. By repeating  
 the same argument until we reach  $v$ , we obtain an assignment of the portions of the  
 interior of  $e$  contained in  $\text{FCVD}^*(S)$  (except for  $e'$ ) to segments in  $M_{\text{bis}}^e$ . Furthermore,  
 in this assignment no segment is charged more than one portion. This completes the  
 proof of the lemma.  $\square$

453 **Theorem 2.** *Let  $S$  be a set of  $n$  segments in the plane in general position. Then*  
 454  $|\text{FCVD}^*(S)| = O(|\text{HVD}(S)| + |\text{FCVD}(S)| + m_{bis}).$

455 *Proof.* Consider a face  $f$  of  $\text{FCVD}^*(S)$ . By Lemma 9,  $f \cap \text{HVD}(S)$  is a connected  
 456 graph. Each vertex of this graph (if any) has degree three, since it is a vertex of  $\text{HVD}(S)$ .  
 457 By Euler's formula the number of h-vertices on  $\partial f$  is  $O(H+1)$ , where  $H$  is the number  
 458 of vertices of  $\text{HVD}(S)$  inside  $f$ . Analogously, the number of fc-vertices on  $\partial f$  is  $O(F+1)$ ,  
 459 where  $F$  is the number of vertices of  $\text{FCVD}(S)$  inside  $f$ .

460 Any vertex on  $\partial f$  that is neither an h-vertex nor an fc-vertex is a mixed vertex of  
 461  $\text{HVD}(S)$  or of  $\text{FCVD}(S)$ . Clearly, each pure vertex of  $\text{HVD}(S)$  and of  $\text{FCVD}(S)$  lies  
 462 in at most one face of  $\text{FCVD}^*(S)$ , and each mixed vertex of one of these diagrams lies  
 463 on the boundary of exactly two faces of  $\text{FCVD}^*(S)$ . Thus the total complexity of the  
 464 boundary of all components of  $\text{FCVD}^*(S)$  that contain at least one vertex of  $\text{HVD}(S)$   
 465 or of  $\text{FCVD}(S)$  is  $O(\text{HVD}(S) + \text{FCVD}(S))$ .

466 What remains is to bound the number of components of  $\text{FCVD}^*(S)$  that do not  
 467 contain a vertex of  $\text{HVD}(S)$  or of  $\text{FCVD}(S)$ . We consider separately the unbounded  
 468 and the bounded components of this type. The total number of unbounded ones is  
 469  $O(|\text{HVD}(S)| + |\text{FCVD}(S)|)$ : By Lemma 11, each such unbounded component con-  
 470 tains an unbounded portion of an edge of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , and clearly one edge  
 471 can correspond to at most one component. By Lemma 11, each bounded component of  
 472  $\text{FCVD}^*(S)$  with no vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$  intersects exactly one pure edge  
 473 of  $\text{HVD}(S)$ , and this intersection is connected. Thus the number of such components of  
 474  $\text{FCVD}^*(S)$  can be upper-bounded by the total number of intersections of all pure edges  
 475 of  $\text{HVD}(S)$  with  $\text{FCVD}^*(S)$ . By Lemma 12, this is  $O(|\text{HVD}(S)| + m_{bis})$ .

Summing up the total complexity of the boundary for all types of components of  
 $\text{FCVD}^*(S)$  implies the claim.  $\square$

## 476 4 Computing $\text{FCVD}^*(S)$

477 Recall from Section 3.2 that any unbounded component of  $\text{FCVD}^*(S)$  corresponds  
 478 to a stabbing line for  $S$ . All the stabbing lines and the corresponding components of  
 479  $\text{FCVD}^*(S)$  can be found in  $O(n \log n)$  time [15]. From now on we assume that  $S$  has  
 480 no stabbing line, and therefore all components and faces of  $\text{FCVD}^*(S)$  are bounded.

### 481 4.1 General algorithm

482 By Lemmas 6, 10 and 11, any bounded face  $f$  of  $\text{FCVD}^*(S)$  has at least one of the  
 483 following properties: (1)  $f$  is incident to a mixed vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , (2)  $f$   
 484 contains a pure vertex of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$ , or (3)  $f$  contains exactly one segment  
 485 of a pure edge of both  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ . Our algorithm, described in Figure 13,  
 486 has three parts. The first part (Steps 1–7) computes the faces of property (1). The second  
 487 part (Steps 8–12) computes the faces of property (2) that have not been computed so  
 488 far. After that the faces that satisfy only property (3) are left, and they are computed in  
 489 Steps 13–14.

490 We next make some remarks about the first two parts of the algorithm in Figure 13.

**Algorithm** COMPUTING  $\text{FCVD}^*(S)$

1. compute  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ ;
2. **for** each non-marked mixed vertex  $v$  of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$  **do**
3.     **if**  $v \in \partial\text{FCVD}^*(S)$  **then**
4.         **for** each face  $f$  of  $\text{FCVD}^*(S)$  incident to  $v$ ;
5.             **if**  $f$  has not been visited yet **then**
6.                 trace  $\partial f$  starting from  $v$ ;
7.                 mark all vertices of  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  in  $\bar{f}$ ;
8.     **for** each non-marked pure vertex  $u$  of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$  **do**
9.         **if**  $u \in \text{FCVD}^*(S)$  **then**
10.             let  $f$  be the face of  $\text{FCVD}^*(S)$  that contains  $u$ ;
11.             find a point on  $\partial f$  and trace  $\partial f$ ;
12.             mark all vertices of  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  in  $\bar{f}$ ;
13.     **for** each segment  $e$  of a pure edge of  $\text{HVD}(S)$  that is outside the computed faces of  $\text{FCVD}^*(S)$  **do**
14.         compute all faces of  $\text{FCVD}^*(S)$  that intersect  $e$ ;

**Fig. 13.** Algorithm to compute  $\text{FCVD}^*(S)$ .

Steps 3 and 9 are simple, after  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  are computed and preprocessed for point location queries. For example, if  $v$  is a vertex of say  $\text{HVD}(S)$ , we first locate  $v$  in  $\text{FCVD}(S)$ . Once we obtain a segment  $aa' \in S$  such that  $v \in \overline{\text{freg}}(aa')$ , we can compare the Hausdorff and farthest-color radii of  $v$  in constant time.

The loop in Step 4 performs exactly two iterations, since each mixed vertex on  $\partial\text{FCVD}^*(S)$  has exactly two incident faces of  $\text{FCVD}^*(S)$  (see Section 3.2).

In Step 11, given a point  $u$  inside face  $f$ , we proceed as follows to find a point on  $\partial f$ . We first locate  $u$  in both  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ . Then we trace in both diagrams the vertical ray originating at  $u$ , until we reach the boundary of  $f$ . To perform this efficiently, we preprocess  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  for ray-shooting queries.

The third part of the algorithm (Steps 13–14) is discussed in Section 4.2, its correctness in Section 4.3, and its time complexity in Section 4.4 (specifically, see Theorem 3).

## 4.2 Searching in a pure edge of $\text{HVD}(S)$

In this section, we present an algorithm to compute all faces of  $\text{FCVD}^*(S)$  intersected by a given portion of a pure edge of  $\text{HVD}(S)$ .

Due to the assumption that  $S$  does not have any stabbing line, all faces of  $\text{FCVD}^*(S)$  we are searching for are bounded. Thus, the input portion of a pure edge of  $\text{HVD}(S)$  is assumed to be a line segment. The algorithm is given as a pseudocode in Figure 15.

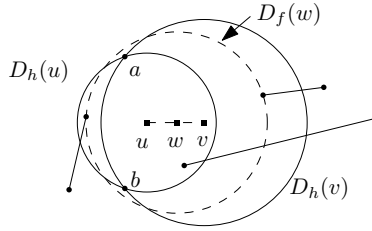
Without loss of generality, we assume that the input segment  $e = uv$  is horizontal, and that  $u$  is to the left of  $v$ . The algorithm assumes that, if  $e$  is shrunk infinitesimally from both sides (see Step 1), the resulting segment  $u_r v_\ell$  has both endpoints outside  $\text{FCVD}^*(S)$ . To proceed with the description of the algorithm, we need to introduce some notation. For convenience, we assume that the segment  $uv$  is already shrunk.

514 Let  $e = uv$  be a line segment on a pure edge of  $\text{HVD}(S)$  separating  $\text{hreg}(a)$  and  
 515  $\text{hreg}(b)$ , such that  $u$  and  $v$  are outside  $\text{FCVD}^*(S)$ . Additionally, if the segment  $ab$  in-  
 516 tersects the interior of  $e$ , this intersection divides  $e$  into two portions, which we process  
 517 separately. Note that neither  $u$  nor  $v$  are mixed vertices of  $\text{HVD}(S)$  (since  $uv$  is the  
 518 result of shrinking a portion of a pure edge of  $\text{HVD}(S)$  from both sides).

519 We classify the segments in  $S \setminus \{aa', bb'\}$  with respect to  $uv$  as segments of types  
 520 *left, right, middle, in, out*, as in Section 3.3 (see Figure 12). Using this classification, for  
 521 any point  $w$  in  $e$ , we define  $\text{type}(w)$  as a set containing one element per each  $cc' \in S$   
 522 such that  $w \in \overline{\text{fcreg}}(cc')$ . The elements of  $\text{type}(w)$  are defined as follows: Let  $cc'$  be  
 523 a segment in  $S$  such that  $w \in \overline{\text{fcreg}}(cc')$ . If  $cc'$  is not of type *middle*, then we add the  
 524 type of  $cc'$  to  $\text{type}(w)$ . If  $cc'$  is of type *middle*, then either  $c$  or  $c'$  (say,  $c$ ) is contained  
 525 in  $D_h(u) \setminus D_h(v)$ , and the other endpoint ( $c'$ ) is contained in  $D_h(v) \setminus D_h(u)$ . We  
 526 further differentiate the classification *middle* as follows: If  $w$  lies on  $\text{bis}(c, c')$ , then  
 527  $mm \in \text{type}(w)$ . Otherwise, if  $w \in \overline{\text{fcreg}}(c)$ , then  $ml \in \text{type}(w)$ ; if  $w \in \overline{\text{fcreg}}(c')$ ,  
 528 then  $mr \in \text{type}(w)$ . When we need to specify  $cc'$ , we do as follows: Imagine that  
 529  $w \in \overline{\text{fcreg}}(cc')$  and  $cc'$  is of type *in*. Then we say  $\text{in} \in \text{type}(w)$  caused by  $cc'$ .

530 Further, we use  $\tilde{l}$  to denote types *left* and  $ml$ , and we use  $\tilde{r}$  to denote *right* and  $mr$ .

531 **Definition 5.** A point  $w$  in  $e$  is a changing point if  $\{\tilde{r}, \tilde{l}\} \subseteq \text{type}(w)$  (see Figure 14).



**Fig. 14.**  $w$  is a changing point (not in  $\text{FCVD}^*(S)$ ).

532 A changing point might or might not be in  $\text{FCVD}^*(S)$ . Note that a changing point  $w$   
 533 is an intersection point between a pure edge of  $\text{HVD}(S)$  and a pure edge of  $\text{FCVD}(S)$ .  
 534 Intuitively, at  $w$  the point giving the farthest-color radius changes from being in  $D_h(v) \setminus$   
 535  $D_h(u)$  to being in  $D_h(u) \setminus D_h(v)$ , i.e.,  $\tilde{r} \in \text{type}(w_\ell)$  and  $\tilde{l} \in \text{type}(w_r)$  (see Figure 14).  
 536 Then we have the following.

537 **Lemma 13.** (a) If  $e$  intersects a face of  $\text{FCVD}^*(S)$  in a segment  $e'$ , then there exists a  
 538 point  $w$  in  $e'$  such that  $\text{in} \in \text{type}(w)$  or  $w$  is a changing point. (b) If there is a point  
 539  $w \in e$  such that  $\text{out} \in \text{type}(w)$ , then  $e$  does not intersect  $\text{FCVD}^*(S)$ . (c) If there is a  
 540 point  $w \in e$  such that  $\text{in} \in \text{type}(w)$ , then  $w \in \text{FCVD}^*(S)$ .

541 *Proof.* Even though item (a) is the base of the algorithm SEARCH IN  $e$ , it is actually  
 542 not needed to prove its correctness, given in Lemma 16, because Lemma 16, together  
 543 with Lemmas 14 and 15, in a way reproves item (a). Therefore, we omit the proof of  
 544 this item.

545 (b) Let  $e = uv$ , and let  $cc'$  be a segment in  $S$  that causes *out* in  $type(w)$ . Then both  
 546  $c$  and  $c'$  are outside  $D_h(u) \cup D_h(v)$ . For any point  $x \in e$ ,  $D_h(x) \subset D_h(u) \cup D_h(v)$ ,  
 547 thus both  $c$  and  $c'$  are outside  $D_h(x)$ , and therefore  $x \notin \text{FCVD}^*(S)$ . This implies that  
 548  $e \cap \text{FCVD}^*(S) = \emptyset$ .

(c) If  $in \in type(w)$ , then there is a segment  $cc' \in S$  such that  $w \in \overline{\text{fcreg}}(cc')$  and  
 one of the endpoints of  $cc'$  is in  $D_h(u) \cap D_h(v)$ . This endpoint is the closest one to  $w$   
 among  $c, c'$ . Thus it is on the boundary of  $D_f(w)$ , and  $D_f(w) \subset D_h(w)$ , which implies  
 the claim.  $\square$

549 Thus, it is enough to examine the changing points of  $e$ , and the points  $w$  such that  
 550  $in \in type(w)$ . To find such points, we use the *find-change query* subroutine, defined  
 551 next. When dealing with a subsegment  $ts$  of  $e$ , or a pair  $(t, s)$  of points in  $e$ , we write  
 552 the left-most point first.

553 **Definition 6 (Find-change query).** *The input of the query is a pair  $(t, s)$  of points in*  
 554  *$e$ , such that  $type(t)$  contains  $\tilde{r}$  but not  $\tilde{l}$ , and  $type(s)$  contains  $\tilde{l}$  but not  $\tilde{r}$ . The query*  
 555 *returns a point  $w$  in the segment  $ts$  such that one of the following holds: (i)  $w$  is a*  
 556 *changing point; (ii)  $in \in type(w)$ ; (iii)  $out \in type(w)$ .*

557 For the sake of clarity, we defer the proof that the find-change query is well-defined,  
 558 as well as the proof of correctness of the remaining pieces of the algorithm, to the next  
 559 subsection.

**Algorithm SEARCH IN  $e = uv$**

1.  $uv \leftarrow u_r v_\ell$ ; (\* shrink the segment  $uv$  infinitesimally \*)
2. **if**  $\tilde{r}$  is the only element in  $type(u)$  **and**  $\tilde{l}$  is the only element in  $type(v)$   
**then**
3.     perform a find-change query on  $(u, v)$ ;
4.     let  $w$  be the point returned by the query;
5.     **if**  $out \in type(w)$  **then**
6.         **return**;
7.     **if**  $w \in \text{FCVD}^*(S)$  **then**
8.         trace  $\partial f$ , where  $f$  is the face of  $\text{FCVD}^*(S)$  that contains  $w$ ;
9.         let  $q, q'$  be the points of  $\partial f \cap uv$ , where  $q$  is to the left of  $q'$ ;
10.     **else** (\*  $w$  is a changing point in  $uv$  and  $w \notin \text{FCVD}^*(S)$  \*)
11.         let  $q, q'$  both be  $w$ ;
12.     SEARCH IN  $uq$ ;
13.     SEARCH IN  $q'v$ ;
14.     **return**;
15. **else return**;

**Fig. 15.** Algorithm to compute all faces of  $\text{FCVD}^*(S)$  intersected by  $e = uv$ .

560 We are now ready to describe our algorithm SEARCH IN  $e$ , which computes all the  
 561 faces of  $\text{FCVD}^*(S)$  intersected by  $e$ . The algorithm is illustrated in Figure 15; it uses

the characterization from Lemma 13. At any time, the algorithm processes a subsegment of  $e$ . It first shrinks  $e$  infinitesimally to ensure that its endpoints do not belong to  $\text{FCVD}^*(S)$  (Step 1). Then (in Step 2) it performs a check that allows to eliminate some execution paths, when it is guaranteed that  $e$  does not intersect  $\text{FCVD}^*(S)$ . If the check is passed, the search continues as follows. It performs a find-change query on  $uv$  that returns a point  $w$  (Steps 3–4). If  $out \in \text{type}(w)$ , the algorithm stops (Step 6), since it is guaranteed that  $e \cap \text{FCVD}^*(S) = \emptyset$  by Lemma 13b. Otherwise, the algorithm proceeds. In case  $w$  is in  $\text{FCVD}^*(S)$ , the face containing  $w$  is traced (Step 8). In both cases ( $w$  is in  $\text{FCVD}^*(S)$  or not), the algorithm calls itself recursively for two disjoint subsegments  $uq$  and  $q'v$  of  $e$  (Steps 12–13). Initially, the algorithm is called for  $e = uv$ .

We remark that the faces of  $\text{FCVD}^*(S)$  intersecting  $e$  are not found in a left-to-right or any other “natural” order, as the find-change query finds *some* point of the desired property. This is the reason why, after finding a changing point  $w \in e$ , the algorithm continues searching on both sides of  $w$ . We remark as well that, at every recursive call of the algorithm, the function  $\text{type}(\cdot)$  is re-defined according to the extremes of the segment on which the procedure is called. Therefore, for a point  $x \in e$ , the value of  $\text{type}(x)$  may change as the algorithm proceeds. Below we will refer to it as “ $\text{type}(x)$  with respect to  $e$ ”, except for the cases when there is no ambiguity.

### 4.3 Correctness

We now prove that the algorithm presented in Sections 4.1 and 4.2 is correct. First we show that Find-change query is well-defined:

**Lemma 14.** *If the pair  $(t, s)$  satisfies the conditions of the input of Find-change query, then there exists a point  $w$  in the segment  $ts$  such that  $w$  is a changing point,  $in \in \text{type}(w)$ , or  $out \in \text{type}(w)$ .*

*Proof.* Suppose for the sake of contradiction that there is no such point  $w$ . Recall that there are five generic types for the points of  $e$ , namely  $\{in, out, \tilde{l}, \tilde{r}, mm\}$ . Since the lemma is not fulfilled, segment  $ts$  can be partitioned into open subsegments of points whose only type is either  $\tilde{r}$  or  $\tilde{l}$ , and two consecutive such subsegments are separated by a point, whose type contains  $mm$  (the leftmost and the rightmost subsegment includes one border point  $t$  and  $s$  respectively, so they are half-open). Therefore, when traveling along  $ts$  from left to right, we encounter two consecutive subsegments of  $ts$  such that the only type of points in the first one is  $\tilde{r}$ , and the only type in the second one is  $\tilde{l}$ . Let  $w'$  be the point that separates these two subsegments;  $mm \in \text{type}(w')$ . We next argue that in this situation  $w'$  is a changing point, which will yield a contradiction.

Suppose that  $mm$  with multiplicity one is the only element in  $\text{type}(w')$  (due to our assumption that no four endpoints of segments in  $S$  are cocircular,  $mm$  cannot be in  $\text{type}(w')$  with multiplicity greater than one). Let  $cc'$  be the segment in  $S$  such that  $mm \in \text{type}(w')$  caused by  $cc'$ . We assume without loss of generality that  $c'$  is contained in  $D_h(u) \setminus D_h(v)$ , and  $c$  in  $D_h(v) \setminus D_h(u)$ . Then  $w' \in \text{fcreg}(cc')$  and the farthest-color disk  $D_f(w')$  has on the boundary  $c$ ,  $c'$ , and no other endpoint of any segment in  $S$ . Consequently,  $w'_\ell \in \text{fcreg}(c')$ . This implies that  $\text{type}(w'_\ell) = \{ml\}$ , which contradicts the fact that the subsegment to the left of  $w'$  has only type  $\tilde{r}$ . Thus,

there is another element in  $\text{type}(w')$  apart from  $mm$ , and this element can only be  $\tilde{r}$ . Arguing analogously with  $w'_r$ , we find that  $\tilde{l} \in \text{type}(w')$ . We conclude that  $w'$  is a changing point, and arrive to a contradiction.  $\square$

596 The following lemma refers to Step 2 of the algorithm SEARCH IN  $e$ :

597 **Lemma 15.** *Let  $e = uv$ , where  $u, v \notin \text{FCVD}^*(S)$ . If there is an element  $x$  in*  
 598  *$\text{type}(u)$  such that  $x \neq \tilde{r}$  or there is an element  $y$  in  $\text{type}(v)$  such that  $y \neq \tilde{l}$ , then*  
 599  *$uv \cap \text{FCVD}^*(S) = \emptyset$ .<sup>5</sup>*

*Proof.* Recall that, if a point  $w$  belongs to  $\text{FCVD}^*(S)$ , then  $D_h(w)$  contains at least one endpoint of every segment in  $S$ . Since  $u \notin \text{FCVD}^*(S)$ , each segment  $cc' \in S$  such that  $u \in \text{fcseg}(cc')$  has both endpoints outside  $D_h(u)$ . Therefore  $in \notin \text{type}(u)$ ,  $\tilde{l} \notin \text{type}(u)$ , and  $mm \notin \text{type}(u)$ . Analogously, since  $v \notin \text{FCVD}^*(S)$ , we have that  $in \notin \text{type}(v)$ ,  $\tilde{r} \notin \text{type}(v)$ , and  $mm \notin \text{type}(v)$ . On the other hand, by Lemma 13b if  $out$  belongs to  $\text{type}(u)$  or to  $\text{type}(v)$ , then  $uv \cap \text{FCVD}^*(S) = \emptyset$ .  $\square$

600 The following lemma proves that the algorithm SEARCH IN  $e$  is correct:

601 **Lemma 16.** *The algorithm SEARCH IN  $e$  computes all faces of  $\text{FCVD}^*(S)$  intersected*  
 602 *by  $e$ .*

603 *Proof.* Let  $e = uv$ . Segment  $uv$  satisfies the input condition of the algorithm: If  $uv$   
 604 is shrunk infinitesimally from both sides, its endpoints are outside  $\text{FCVD}^*(S)$ . This  
 605 follows from Steps 13–14 of the algorithm COMPUTING  $\text{FCVD}^*(S)$ .

606 The condition checked in Step 2 of the algorithm is justified by Lemma 15: If the  
 607 condition is not satisfied, then  $uv \cap \text{FCVD}^*(S) = \emptyset$  and we can safely stop the search  
 608 at Step 15. Otherwise, the input conditions of the find-change query are satisfied.

609 By Lemma 14, the find-change query is well-defined, and there are three possible  
 610 outputs when we perform it on  $(u, v)$ . If the query returns a point  $w$  such that  $out \in$   
 611  $\text{type}(w)$ , then  $e \cap \text{FCVD}^*(S) = \emptyset$  due to Lemma 13b. In this case, the algorithm stops  
 612 (see Steps 5–6). Otherwise, a face of  $\text{FCVD}^*(S)$  containing  $w$  is traced if and only if  
 613  $w \in \text{FCVD}^*(S)$ .

614 The algorithm is called recursively for two subsegments of  $uv$ , which are  $uq$  and  
 615  $q'v$  (Steps 12–13). We now show that each of these subsegments satisfies the input  
 616 condition of the algorithm. Indeed, points  $u_r$  and  $v_\ell$  are outside  $\text{FCVD}^*(S)$  because  $uv$   
 617 was satisfying this condition in the first place. Points  $q$  and  $q'$  are determined in Step 9 or  
 618 Step 11, depending on the condition in Step 7. In particular, if the condition is satisfied,  
 619 i.e.,  $w$  lies in a face  $f$  of  $\text{FCVD}^*(S)$ , then  $e \cap f$  is a line segment (see Lemma 11).  
 620 In this case, Step 9 is executed, and it assigns to  $q$  and  $q'$  the left and right endpoints  
 621 of the segment  $e \cap f$ , respectively. This implies that  $q_\ell$  and  $q'_r$  are outside  $\text{FCVD}^*(S)$ .  
 622 Otherwise, Step 11 assigns both  $q, q'$  to  $w$  and, since  $w$  is not in  $\text{FCVD}^*(S)$ , neither are  
 623  $w_\ell, w_r$ .

In Theorem 3, we analyze the running time of the algorithm, and in particular we  
 prove that the algorithm terminates.  $\square$

<sup>5</sup> Note that it is not always possible to shrink  $uv$  infinitesimally so that  $\text{type}(u)$  and  $\text{type}(v)$  consist of one element each. In particular, this is not possible when  $uv$  lies on an edge of  $\text{FCVD}(S)$ , and this situation does not contradict our general position assumption.

#### 624 4.4 Running time

625 Below we analyze the time complexity of the algorithm to compute  $\text{FCVD}^*(S)$ , and  
 626 thus the time complexity to solve the stabbing circle problem.

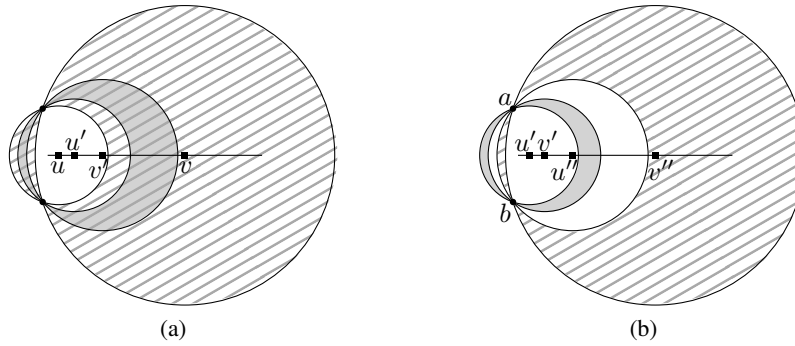
627 We start with the following result concerning the find-change query.

628 **Lemma 17.** *A find-change query can be performed in  $O(\log^2 n)$  time.*

629 *Proof.* For a pair  $(t, s)$  of points, we perform the find-change query as follows. We use  
 630 a point-location data structure for  $\text{FCVD}(S)$ , such that the point location for a query  
 631 point  $q$  is performed by a sequence of  $O(\log n)$  atomic questions of the form “is  $q$   
 632 above or below (respectively, to the left or right of) a line  $\ell$ ?” (e.g., the data structure by  
 633 Edelsbrunner et al. [17], or the one by Kirkpatrick [20]). Notice that, in our case, instead  
 634 of a fixed point  $q$ , we only have a pair  $(t, s)$  such that the segment  $ts$  contains a sought  
 635 point (a changing point, a point whose type contains *in* or a point whose type contains  
 636 *out*). An atomic question is processed as follows. If  $ts \cap \ell = \emptyset$ , the answer is the same  
 637 for any point in  $ts$ , and we continue with the pair  $(t, s)$ . Otherwise, let point  $p$  be  $ts \cap \ell$ .  
 638 First, the standard point location for  $p$  in  $\text{FCVD}(S)$  gives us  $\text{type}(p)$ . If  $\tilde{r} \in \text{type}(p)$   
 639 and  $\tilde{l} \notin \text{type}(p)$ , we continue with the pair  $(p, s)$ . Symmetrically, if  $\tilde{l} \in \text{type}(p)$  and  
 640  $\tilde{r} \notin \text{type}(p)$ , we continue with the pair  $(t, p)$ . If  $\text{type}(p) = \{mm\}$ , then we continue  
 641 with either  $(p, s)$  or  $(t, p)$ , because they both satisfy the input condition of the query.  
 642 Otherwise, we stop the procedure, and return  $p$ . Clearly, this happens in one of the  
 643 following cases: (i)  $\{\tilde{l}, \tilde{r}\} \subseteq \text{type}(p)$ ; (ii)  $\text{in} \in \text{type}(p)$ ; or (iii)  $\text{out} \in \text{type}(p)$ .

Answering one atomic question within the procedure takes  $O(\log n)$  time, and the  
 whole find-change query takes  $O(\log^2 n)$  time.

A similar idea of simulating a point location for an unknown point is used in  
 Cheong et al. [9], and in Cheilaris et al. [8, Section 7].  $\square$



**Fig. 16.** Illustration for the proof of Lemma 18.

644 **Lemma 18.** *Let  $e$  be a segment on a pure edge of  $\text{HVD}(S)$ , and  $cc'$  be a segment in  $S$ .*

645 (a) *If  $cc'$  is of type middle for a segment  $e' \subset e$ , then  $cc'$  is of type middle for  $e$ .*



646 (b) Let  $e', e''$  be two disjoint subsegments of  $e$ . Then  $cc'$  is of type middle for at most  
 647 one of  $e', e''$ .

648 *Proof.* Let  $e = uv$ ,  $e' = u'v'$ , and  $e'' = u''v''$ .

- 649 (a) If  $cc'$  is of type middle for  $e'$ , then one of its endpoint is in  $D_h(u') \setminus D_h(v')$  and  
 650 the other one is in  $D_h(v') \setminus D_h(u')$  (see the two gray areas in Figure 16a). These  
 651 areas are contained in  $D_h(u) \setminus D_h(v)$  and  $D_h(v) \setminus D_h(u)$ , respectively (shown with  
 652 tiling pattern in Figure 16a).  
 653 (b) Suppose  $cc'$  is of type middle for both  $e'$  and  $e''$ . One of the endpoints of  $cc'$  lies  
 654 in  $D_h(u') \setminus D_h(v')$  and the other one in  $D_h(v') \setminus D_h(u')$ . Analogously, one of its  
 655 endpoints lies in  $D_h(u'') \setminus D_h(v'')$  and the other one in  $D_h(v'') \setminus D_h(u'')$ . But  
 656 these four areas are disjoint, see shaded and tiled areas in Figure 16b (note that the  
 657 endpoints  $a, b$  of two segments in  $S$  such that  $e \subset bis(a, b)$  belong to neither of  
 658 these four areas since all the disks  $D_h(\cdot)$  are closed). We arrive to a contradiction.

□

659 We now prove that the running time of SEARCH IN  $e$  is related to the number of  
 660 segments of  $S$  of type middle for  $e$ , denoted by  $m_e$ .

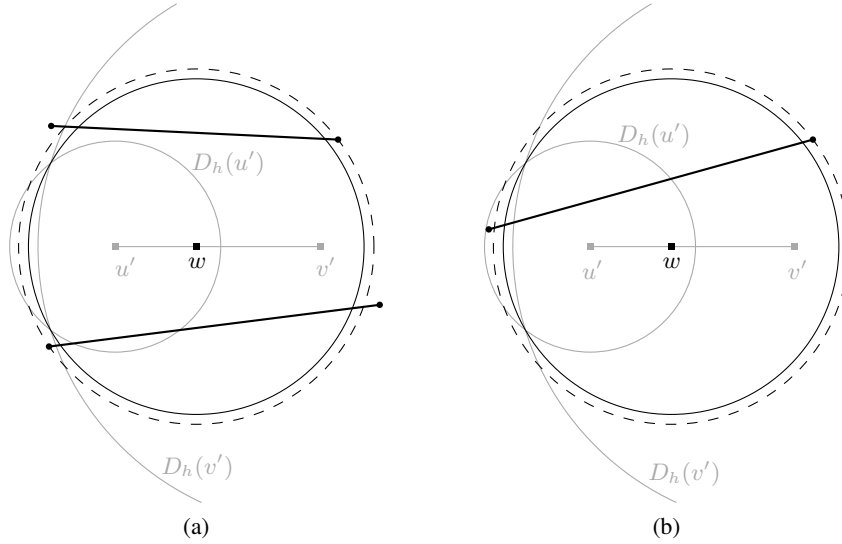
661 **Lemma 19.** *Let  $e$  be a segment on a pure edge of  $HVD(S)$ , such that the algorithm*  
 662 *SEARCH IN is called for  $e$ . Then number of recursive calls performed by the algorithm*  
 663 *SEARCH IN  $e$  is  $O(1 + m_e)$ .*

664 *Proof.* Let  $e = uv$ ; assume without loss of generality that  $e$  is horizontal and  $u$  is its left  
 665 endpoint. Consider the recursion tree of the algorithm SEARCH IN  $e$ . Observe that this  
 666 tree is a proper binary tree (each call of the algorithm causes either zero or two recursive  
 667 calls). Thus the total number of its nodes is linear in the number of its intermediate  
 668 nodes. Therefore, it remains to prove that the number of intermediate nodes of the  
 669 recursion tree of SEARCH IN  $e$  is  $O(1 + m_e)$ .

670 We will charge each intermediate node of the recursion tree to an endpoint of some  
 671 segment of type middle for  $e$ , and each such endpoint will be charged at most once.  
 672 Our charging scheme is symmetric: the nodes that are left children of their parents are  
 673 charged to right endpoints, and vice versa. Thus we will discuss the charging scheme  
 674 for the former case (left children) only. We base on the fact that for each segment of  
 675 type *middle* for  $e$  there is at most one moment during the course of the algorithm when  
 676 this segment becomes of type *left*. Formally, for any segment  $cc' \in S$  that is of type  
 677 *middle* for  $e$ , there is at most one node  $N$  in the recursion tree of the algorithm, such  
 678 that for the subsegment of  $e$  that corresponds to  $N$ ,  $cc'$  is of type *middle*, and for the  
 679 subsegment of  $e$  that corresponds to the left child of  $N$ ,  $cc'$  is of type *left*. This is easy  
 680 to see from the following simple observations. First, if  $cc'$  has stopped being middle  
 681 for the left child of  $N$ , then  $cc'$  stops being middle for the right child of  $N$  as well (it  
 682 becomes of type *right*). Further, if there were two such nodes  $N$  and  $N'$  for which  
 683 the above property would hold, then  $N$  and  $N'$  would belong to one root-to-leaf path in  
 684 the recursion tree (otherwise, the subsegments of  $e$  corresponding to  $N$  and  $N'$  would  
 685 be disjoint, and due to Lemma 18b  $cc'$  could not be of type middle for both of them).  
 686 Suppose that  $N$  is an ancestor of  $N'$ . On one root-to-leaf path, the segment on which

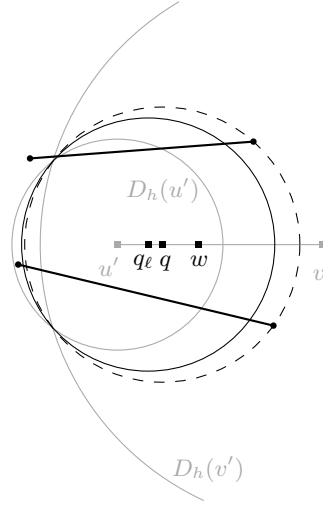
687 the algorithm is called only shrinks and never expands, and thus by Lemma 18a after  
 688  $cc'$  has stopped being middle it never can be middle again (in particular, it cannot be  
 689 middle for the segment corresponding to  $N'$ ). We charge the left child of such node  $N$   
 690 to the right endpoint of  $cc'$ . We proceed with the technical details.

691 Consider a non-leaf node  $N$  of the recursion tree; let  $u'v'$  be the subsegment of  $e$  that  
 692 corresponds to node  $N$ . Since  $N$  is not a leaf, the algorithm has found a point  $w \in u'v'$   
 693 such that either (1)  $w \notin \text{FCVD}^*(S)$ ,  $w$  is a changing point, and  $out \notin \text{type}(w)$ , or (2)  
 694  $w \in \text{FCVD}^*(S)$ . We consider these two cases separately.



**Fig. 17.** Illustration for the proof of Lemma 19: the case when  $w \notin \text{FCVD}^*(S)$  (case 1). Disks  $D_h(u')$ ,  $D_h(v')$  (gray lines),  $D_h(w)$  (black solid lines);  $D_f(w)$  (black dashed lines). (a)  $right \in \text{type}(w)$  caused by  $cc'$ : two possibilities for the segment  $cc'$  (bold line segment):  $w \in \text{fcreg}(c')$  (above), and  $w \in \text{fcreg}(c)$  (below). (b)  $mr \in \text{type}(w)$  caused by  $cc'$ : a placement of  $cc'$  (bold line segment).

695 *Case 1:  $w \notin \text{FCVD}^*(S)$ ,  $w$  is a changing point, and  $out \notin \text{type}(w)$  (see Figure 17).*  
 696 The two children of the node  $N$  are the recursive calls for  $u'w$  and for  $wv'$  respectively,  
 697 and we will consider only the former one (the left child of  $N$ ). Let  $cc'$  be the segment in  
 698  $S$  that causes  $\text{type } \tilde{r}$  for  $w$  with respect to  $u'v'$ ; and let  $c'$  be the endpoint of  $cc'$  such that  
 699  $c' \in D_h(v') \setminus D_h(u')$ . There are two possibilities:  $cc'$  causes  $\text{type } right$ , or it causes  $\text{type } mr$   
 700 (both types are defined with respect to  $u'v'$ ); see respectively Figures 17a and 17b. In  
 701 the former case, the call of the algorithm for  $u'w$  corresponds to a leaf of the recursion  
 702 tree: indeed,  $out \in \text{type}(w_\ell)$  with respect to  $u'_r w_\ell$  caused by  $cc'$  (see Figure 17a), and  
 703 the algorithm will stop at Line 2. Thus no charging is needed for the left child of node  
 704  $N$  in this case. Figure 17a shows two possible placements of  $cc'$ , for both of which the  
 705 argument holds.



**Fig. 18.** Illustration for the proof of Lemma 19, the case when  $w \in \text{FCVD}^*(S)$  (case 2). Disks  $D_h(u')$ ,  $D_h(v')$  (gray lines),  $D_h(q) = D_f(q)$  (black dashed lines);  $D_h(q_\ell)$  (black solid lines). Two possibilities for the segment  $cc'$  (bold line segment).

706 Consider the latter case, i.e., when  $cc'$  causes type  $mr$  with respect to  $u'v'$ . Observe  
 707 that this may only happen when  $w \in \text{fcreg}(c')$ : otherwise,  $cc'$  would by definition  
 708 cause type  $ml$  or  $mm$ , rather than  $mr$ . This implies that  $cc'$  is of type *left* for  $u'_r w_\ell$ , see  
 709 Figure 17b. We charge the call of the algorithm for segment  $u'w$  (i.e., the left child of  
 710 node  $N$ ) to the endpoint  $c'$  of  $cc'$ .

*Case 2:  $w \in \text{FCVD}^*(S)$  (see Figure 18).* The left child of node  $N$  in the recursion tree is the call of the algorithm for subsegment  $u'q$  of  $u'v'$ , where  $q$  is the first point of the boundary of  $\text{FCVD}^*(S)$  that is encountered when traversing  $e$  from  $w$  in the left direction. Since  $q \in \text{FCVD}^*(S)$ , each segment in  $S$  has at least one endpoint inside  $D_h(q)$ . Let  $cc'$  be a segment in  $S$  such that  $q_\ell \in \text{fcreg}(cc')$ ; specifically let  $c'$  be the endpoint of  $cc'$  such that  $q_\ell \in \text{fcreg}(c')$ . Since  $q$  belongs to the boundary of  $\text{FCVD}^*(S)$ ,  $q_\ell \notin \text{FCVD}^*(S)$ , and thus both  $c$  and  $c'$  are outside  $D_h(q_\ell)$ . This implies that  $c'$  lies in the portion of the boundary of  $D_h(q)$  contained in  $D_h(v')$ , and  $c$  lies outside  $D_h(q)$  (see Figure 18). Now, there are again two possibilities:  $cc'$  is of type *right* for  $u'v'$ , or  $cc'$  is of type *middle* for  $u'v'$ . In the former case,  $cc'$  is of type *out* for  $u'_r q_\ell$ , thus the call of the algorithm on  $u'q$  is a leaf in the recursion tree, and no charging is needed for the left child of node  $N$ . Otherwise, i.e. if  $cc'$  is of type *middle* for  $u'v'$ ,  $cc'$  is of type *left* for  $u'_r q_\ell$ , and we charge the call of the algorithm on  $u'q$  (the left child of node  $N$ ) to  $c'$ .  $\square$

711 The next lemma allows to bound the total time of all executions of Step 11 of COM-  
 712 PUTING  $\text{FCVD}^*(S)$ :

713 **Lemma 20.** *Let  $f_1, f_2, \dots, f_k$  be the faces of  $\text{FCVD}^*(S)$ . The total number of edges of*  
 714 *HVD( $S$ ) and  $\text{FCVD}(S)$  intersected by these faces is  $O(k + |\text{HVD}(S)| + |\text{FCVD}(S)|)$ .*

*Proof.* Let  $I$  denote the total sum  $\sum_{i=1}^k I(f_i)$ , where  $I(f_i)$  denotes the number of edges of  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  that are intersected by  $f_i$ . We need to show that  $I = O(k + |\text{HVD}(S)| + |\text{FCVD}(S)|)$ . By Lemma 9, for each face  $f_i$ ,  $f_i \cap \text{HVD}(S)$  is a connected component. Such connected component can be either (1) a portion of a single edge of  $\text{HVD}(S)$ ; or (2) a component containing portions of at least two edges. Depending on this, we call face  $f_i$  a type-1 face, or a type-2 face respectively. Let  $t$  be the number of type-1 faces, and  $r$  be the number of type-2 faces; then  $t + r = k$ . Clearly, all type-1 faces contribute  $t$  to the total sum  $I$ . Further, one edge of  $\text{HVD}(S)$  intersects at most two type-2 faces. Thus all type-2 faces contribute at most  $2 * |\text{HVD}(S)|$  to  $I$ . Since  $t \leq k$ ,  $\text{HVD}(S)$  contributes at most  $k + 2 * |\text{HVD}(S)|$  to  $I$ . By an analogous argument,  $\text{FCVD}(S)$  contributes at most  $k + 2 * |\text{FCVD}(S)|$  to  $I$ . The claim follows.  $\square$

We are finally ready to prove the main theorem of this section.

Let  $m$  denote the number of pairs formed by a segment  $aa' \in S$  and a pure edge  $e$  of  $\text{HVD}(S)$  such that  $aa'$  is of type *middle* for  $e$ . That is,  $m$  is the total sum of  $m_e$  for all pure edges  $e$  of  $\text{HVD}(S)$ . Let  $\mathcal{T}_{\text{HVD}(S)}$  and  $\mathcal{T}_{\text{FCVD}(S)}$  denote the time to compute  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ , respectively. Let  $\mathcal{T}_q$  denote the time to answer the find-change query. In this paper,  $\mathcal{T}_q$  is  $O(\log^2 n)$ .

**Theorem 3.** *Let  $S$  be a set of  $n$  segments in the plane in general position. Then,  $\text{FCVD}^*(S)$  can be computed in time  $O(\mathcal{T}_{\text{HVD}(S)} + \mathcal{T}_{\text{FCVD}(S)} + (|\text{HVD}(S)| + |\text{FCVD}(S)| + m)(\log n + \mathcal{T}_q))$ .*

*Proof.* Computing  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  (Step 1 of COMPUTING  $\text{FCVD}^*(S)$ ) requires time  $\mathcal{T}_{\text{HVD}(S)} + \mathcal{T}_{\text{FCVD}(S)}$ .

After computing the diagrams, in time  $O(|\text{HVD}(S)| \log n)$  and  $O(|\text{FCVD}(S)| \log n)$  we can preprocess them to answer point-location queries in  $O(\log n)$  time [17,20]. Then Steps 3 and 9 of the algorithm require logarithmic time (see Section 4.1).

We also preprocess  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  to answer ray-shooting queries in  $O(\log n)$  time. The preprocessing can be done in time  $O(|\text{HVD}(S)| \log n)$  and  $O(|\text{FCVD}(S)| \log n)$ , respectively [7]. After such preprocessing, Step 11 requires  $O((1 + I(f)) \log n)$  time, where  $I(f)$  is the number of edges of  $\text{HVD}(S)$  and  $\text{FCVD}(S)$  intersected by  $f$  (see Section 4.1 for more details on Step 11). By Lemma 20, the total time spent for Step 11 is  $O((|\text{HVD}(S)| + |\text{FCVD}(S)| + |\text{FCVD}^*(S)|) \log n)$ .

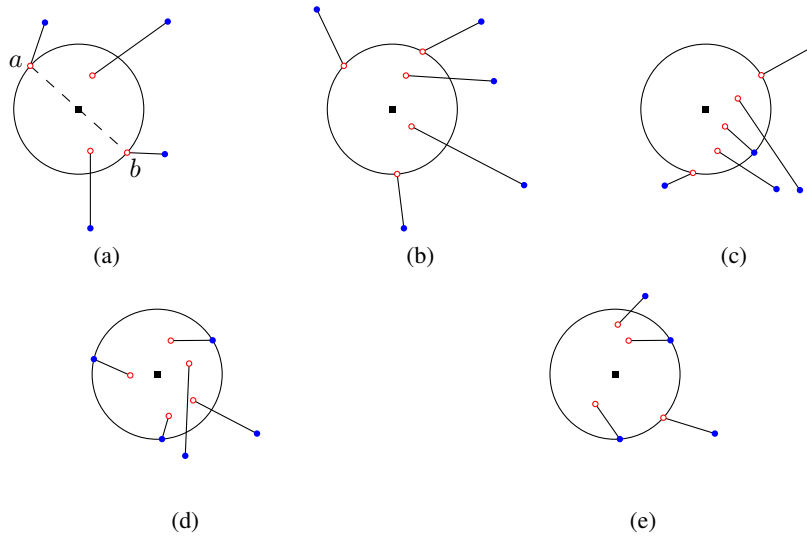
After a point on  $\partial f$  is known,  $\partial f$  can be traced in  $\text{HVD}(S)$  and  $\text{FCVD}(S)$ . Thus the total time required for tracing the boundary of all the faces of  $\text{FCVD}^*(S)$  (in Steps 6 and 11 of COMPUTING  $\text{FCVD}^*(S)$ , and Step 8 of SEARCH IN  $e$ ) is  $O((|\text{HVD}(S)| + |\text{FCVD}(S)| + |\text{FCVD}^*(S)|) \log n)$ .

Therefore, the total time complexity of Steps 2–12 of COMPUTING  $\text{FCVD}^*(S)$  is  $O((|\text{HVD}(S)| + |\text{FCVD}(S)| + |\text{FCVD}^*(S)|) \log n)$ .

The third loop (Steps 13–14) of COMPUTING  $\text{FCVD}^*(S)$  performs  $O(|\text{HVD}(S)| + |\text{FCVD}(S)|)$  iterations. Indeed, the number of faces computed in Steps 2–12 is  $O(|\text{HVD}(S)| + |\text{FCVD}(S)|)$ , and each face has connected intersection with  $\text{HVD}(S)$  by Lemma 9.

In Step 14, SEARCH IN  $e$  is called. By Lemma 19, the total number of recursive calls caused by SEARCH IN  $e = uv$  is  $O(1 + m_e)$ . Summing up  $m_e$  for all pieces  $e$  of edges

of  $\text{HVD}(S)$  on which the procedure is called results in  $O(m)$  due to Lemma 18b. Since any face of  $\text{FCVD}^*(S)$  computed by  $\text{SEARCH IN } e$  intersects only one edge of  $\text{HVD}(S)$  and one edge of  $\text{FCVD}(S)$  and its boundary has constant complexity (see Lemma 11), Step 8 requires constant time. Therefore, one execution of  $\text{SEARCH IN } e$ , except for the recursive calls, is dominated by the find-change query, and thus requires  $O(\log^2 n)$  time (see Lemma 17). Thus the total time required by  $\text{SEARCH IN } e$  is  $O((1 + m_e) \log^2 n)$ . This globally amounts to  $O((|\text{HVD}(S)| + |\text{FCVD}(S)| + m) \log^2 n)$  time for Steps 13–14 of  $\text{COMPUTING FCVD}^*(S)$ . More precisely, this time is  $O((|\text{HVD}(S)| + |\text{FCVD}(S)| + m)(\mathcal{T}_q + \log n))$ .  $\square$



**Fig. 19.** All possibilities for the stabbing circles of minimum and maximum radius.

746 The next lemma shows how to find the largest and smallest stabbing circles, once  
747  $\text{FCVD}^*(S)$  is known. We observe that, in some cases, the stabbing circle of minimum  
748 radius does not exist, because any stabbing circle can be shrunk by decreasing its radius  
749 or slightly moving its center. Moreover, the “limit” circle is not stabbing because the  
750 closed disk induced by the circle contains both endpoints of a segment in  $S$  (see Case 1c  
751 of the proof of Lemma 21 below and Figure 19c). Similarly, it is easy to see that the  
752 stabbing circle of maximum radius never exists, since any stabbing circle can be slightly  
753 enlarged, but the “limit” circle is not stabbing (see Case 2 of the proof of Lemma 21  
754 and Figure 19d,e). In these cases, even though these circles are not stabbing, to simplify  
755 the notation we call these “limit” circles the *stabbing circles with infimum or supremum*  
756 *radius*.

757 **Lemma 21.** *After computing  $\text{FCVD}^*(S)$ , the stabbing circles of minimum (or infi-*  
758 *imum) and supremum radius can be determined in time  $O((|\text{FCVD}^*(S)| + |\text{HVD}(S)| +$   
759  $|\text{FCVD}(S)|) \log n)$ .*

760 *Proof.* We list all the possibilities for such circles below; see Figure 19.

761 Given a segment  $aa'$  and a circle  $c$ , we call “red” the endpoint of  $aa'$  which is closer  
 762 to the center of  $c$ , and we call “blue” the other endpoint (ties are broken arbitrarily).  
 763 Notice that, if  $c$  is stabbing, then the red endpoint of  $aa'$  lies inside the closed disk  
 764 induced by  $c$ , and the blue endpoint lies outside. Standard geometric arguments show  
 765 the following.

- 766 1. The stabbing circle of minimum or infimum radius is a circle passing through:
  - 767 (a) Two red points that are diametrically opposite. In this case the center of the  
 768 circle is at the intersection point between an edge of  $\text{FCVD}(S)$  separating two  
 769 regions  $\text{fcreg}(a)$  and  $\text{fcreg}(b)$  (for some  $aa', bb' \in S$ ), and the segment con-  
 770 necting  $a$  to  $b$ , see Figure 19a. This is a stabbing circle of minimum radius.
  - 771 (b) Three red points. The center is at a vertex of  $\text{FCVD}(S)$ . This is also a stabbing  
 772 circle of minimum radius.
  - 773 (c) Two red points and one blue point. The center is on the boundary of  
 774  $\text{FCVD}^*(S)$ . This is a stabbing circle of infimum radius.
- 775 2. The stabbing circle of supremum radius is a circle passing through:
  - 776 (a) Three blue points. In this case the center of the circle is at a vertex of  $\text{HVD}(S)$ .
  - 777 (b) Two blue points and one red point. The center is on the boundary of  
 778  $\text{FCVD}^*(S)$ .

The stabbing circles with minimum (or infimum) and supremum radius can thus  
 be found by checking the vertices of  $\text{HVD}(S)$  or  $\text{FCVD}(S)$  lying in  $\text{FCVD}^*(S)$ , the  
 edges of  $\text{FCVD}(S)$  intersecting  $\text{FCVD}^*(S)$ , and the boundary of  $\text{FCVD}^*(S)$ . Since in  
 all cases the radius of the corresponding circle can be computed in  $O(\log n)$  time, the  
 claim follows.  $\square$

779 As a conclusion, we put together the above results and connect them with the stab-  
 780 bing circle problem as stated in Section 1. By Lemma 7, distinct faces of  $\text{FCVD}^*(S)$   
 781 correspond to combinatorially different stabbing circles. Thus, the above results yield  
 782 the following:

783 **Corollary 3.** *Let  $S$  be a set of  $n$  segments in the plane in general position. All the*  
 784 *combinatorially different stabbing circles for  $S$ , and the ones with minimum and maxi-*  
 785 *imum radius, can be computed in  $O(\mathcal{T}_{\text{HVD}(S)} + \mathcal{T}_{\text{FCVD}(S)} + (|\text{HVD}(S)| + |\text{FCVD}(S)| +$   
 786  $m) \log^2 n)$  time.*

## 787 5 Parallel Segments

788 Let  $S$  be a set of parallel segments. The goal of this section is to prove the following  
 789 theorem.

790 **Theorem 4.** *The stabbing circle problem for a set  $S$  of  $n$  parallel segments in general*  
 791 *position can be solved in  $O(n \log^2 n)$  time and  $O(n)$  space.*

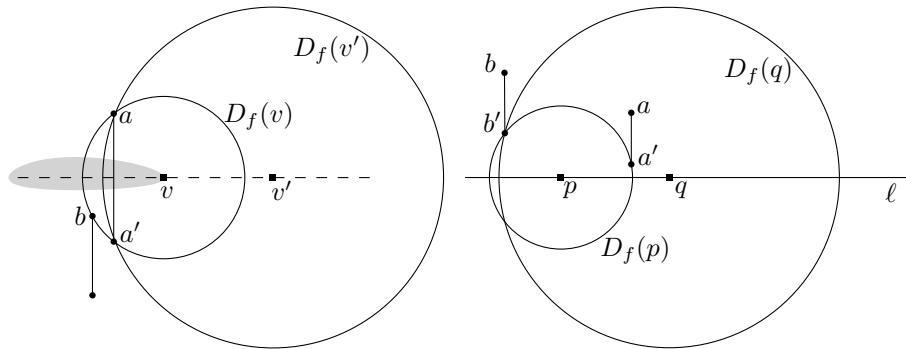
792 We remark that the  $O(\log^2 n)$  factor in the time complexity of Theorem 4 is con-  
 793 tributed by the find-change query. If this query can be answered in  $O(\log n)$  time then  
 794 the stabbing circle problem for  $n$  parallel segments can be solved in  $O(n \log n)$  time.

795 We prove Theorem 4 using Theorem 3 and Corollary 3. Since the segments in  $S$   
 796 are parallel, they must be pairwise disjoint, and thus  $\text{HVD}(S)$  is an instance of *ab-*  
 797 *stract Voronoi diagrams*; hence,  $|\text{HVD}(S)|$  is  $O(n)$  and  $\mathcal{T}_{\text{HVD}(S)}$  is  $O(n \log n)$  [21]. In  
 798 Section 5.1 we show that  $|\text{FCVD}(S)|$  is also  $O(n)$  and  $\mathcal{T}_{\text{FCVD}(S)}$  is  $O(n \log n)$ . In Sec-  
 799 tion 5.2 we show that  $m = O(n)$ . Thus, the algorithm of Section 4 for this particular  
 800 case has time complexity  $O(n(\log n + \mathcal{T}_q))$ , and since  $\mathcal{T}_q$  is  $O(\log^2 n)$  (see Lemma 17)  
 801 we derive Theorem 4.

## 802 5.1 The farthest-color Voronoi diagram for a set of parallel segments

803 **Lemma 22.** *If all segments in  $S$  are parallel, then for each  $aa' \in S$ ,  $\text{bis}(a, a')$  con-*  
 804 *tributes at most one internal edge to  $\text{FCVD}(S)$ .*

*Proof.* Suppose for the sake of contradiction that there are two internal edges  $e$  and  $e'$  in  $\text{FCVD}(S)$  which are portions of  $\text{bis}(a, a')$ . Assume without loss of generality that the segments in  $S$  are vertical, thus,  $\text{bis}(a, a')$  is horizontal, and  $e$  is to the left of  $e'$ . Let  $v, v' \in \text{bis}(a, a')$  be the mixed vertices that are respectively the right endpoint of  $e$  and the left endpoint of  $e'$ . The farthest-color disk  $D_f(v)$  contains an endpoint of every segment in  $S$ , and  $\partial D_f(v)$  passes through points  $a, a'$  and an endpoint  $b$  of some segment  $bb' \in S$ . Observe that  $bb'$  is to the left of  $aa'$ , and  $bb' \cap D_f(v) = \{b\}$  (see Figure 20 (left), where  $\text{fcreg}(aa')$  is shown shaded, and  $\text{bis}(a, a')$  is shown dashed). Since  $v'$  is to the right of  $v$ , the closed disk centered at  $v'$  with radius  $d(v', a)$  does not contain  $b$  nor  $b'$ , thus,  $v' \notin \text{fcreg}(aa')$ . We obtain a contradiction.  $\square$



**Fig. 20.** Left: Illustration for the proof of Lemma 22. Right: Illustration for the proof of Lemma 24.

805 **Lemma 23.** *If all segments in  $S$  are parallel, the structural complexity of  $\text{FCVD}(S)$  is*  
 806  *$O(n)$ .*

*Proof.* By Lemma 5, the number of unbounded faces of  $\text{FCVD}(S)$  is  $O(n)$ . Since all segments in  $S$  are parallel, by Lemma 22, the number of all internal edges in  $\text{FCVD}(S)$  is at most  $n$ . By Lemma 3, every bounded face of  $\text{FCVD}(S)$  has an internal edge on its boundary, thus there are at most  $2n$  bounded faces in  $\text{FCVD}(S)$ . Thus,  $\text{FCVD}(S)$  is a planar graph with  $O(n)$  faces whose vertices have degree three. By Euler's formula,  $|\text{FCVD}(S)| = O(n)$ .  $\square$

**Lemma 24.** *If all segments in  $S$  are parallel, then  $\text{FCVD}(S)$  can be computed in  $O(n \log n)$  time.*

*Proof.* Assume without loss of generality that the segments in  $S$  are vertical. We use the divide-and-conquer technique: We divide  $S$  by a vertical line into two halves,  $S_{\text{left}}$  and  $S_{\text{right}}$ , and recursively compute  $\text{FCVD}(S_{\text{left}})$  and  $\text{FCVD}(S_{\text{right}})$ . Below we prove that the merge curve between  $\text{FCVD}(S_{\text{left}})$  and  $\text{FCVD}(S_{\text{right}})$  is  $y$ -monotone. Such a  $y$ -monotone merge curve can be constructed in  $O(n)$  time by standard arguments on the divide-and-conquer construction of Voronoi diagrams, see e.g., [5]. The claim follows.

Suppose for the sake of contradiction, that the merge curve is not  $y$ -monotone, that is, there are two points  $p, q$  on the merge curve with the same  $y$ -coordinate. Since  $p$  and  $q$  lie on the merge curve, the farthest-color radius of  $p$  with respect to  $S_{\text{left}}$  equals the farthest-color radius of  $p$  with respect to  $S_{\text{right}}$ , and the same holds for  $q$ . Let  $\ell$  be the horizontal line through  $p$  and  $q$ . We redefine  $p$  and  $q$  as the left-most and second left-most points on  $\ell$  that lie on the merge curve, respectively. In  $\text{FCVD}(S)$ , the point at minus infinity on  $\ell$  lies in the farthest-color region of a segment from  $S_{\text{right}}$ . Thus  $p$  lies on the boundary between  $\text{fcreg}(aa')$  (to the left of  $p$ ) and  $\text{fcreg}(bb')$  (to the right of  $p$ ) such that  $aa' \in S_{\text{right}}$  and  $bb' \in S_{\text{left}}$ ; see Figure 20 (right). Then the farthest-color disk  $D_f(q)$  intersects or touches  $bb'$ , and touches a segment  $cc' \in S_{\text{right}}$ . But in this case  $cc'$  is outside the farthest-color disk of  $p$ . We obtain a contradiction.  $\square$

## 5.2 Segments of type middle for a set of parallel segments

The key result of this subsection is the following:

**Lemma 25.** *A segment  $gg' \in S$  is of type middle for at most one pure edge of  $\text{HVD}(S)$ .*

We first prove an easy property of the segments of type middle:

**Lemma 26.** *Suppose that all segments in  $S$  are vertical. Let  $e$  be a pure edge of  $\text{HVD}(S)$  in the boundary of  $\text{hreg}(a)$  and  $\text{hreg}(b)$ , for two segments  $aa', bb' \in S$ . Suppose that segment  $gg' \in S$  is of type middle for  $e$ . Then:*

- (a)  $x(a) \neq x(b)$ .
- (b)  $\min\{x(a), x(b)\} < x(g) < \max\{x(a), x(b)\}$ .

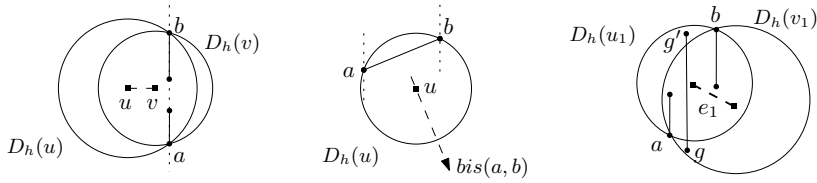
*Proof.* To prove (a), we suppose for the sake of contradiction that  $x(a) = x(b)$ . Then  $e$  is horizontal. Let  $u$  and  $v$  be the left and right endpoints of  $e$ . Since the segment  $gg'$  is of type middle, it has one endpoint to the left of the line  $x = x(a)$ , in  $D_h(u) \setminus D_h(v)$ , and the other endpoint to the right of the line  $x = x(a)$ , in  $D_h(v) \setminus D_h(u)$  (see Figure 21, left). This contradicts the fact that  $gg'$  is vertical.



829 To prove (b), we suppose without loss of generality that  $x(a) < x(b)$ . Then  $e$  is not  
830 horizontal, and we denote by  $u$  and respectively  $v$  the top and bottom endpoints of  $e$ .

831 For any disk  $D$ , we can divide its boundary  $\partial D$  into the left-most point of  $\partial D$ , the  
832 open circular arc containing the upper half portion of  $\partial D$  (called *top chain*), the right-  
833 most point of  $\partial D$ , and the open circular arc containing the lower half portion of  $\partial D$   
834 (called *bottom chain*). Since  $aa'$  is vertical and  $D_h(u)$  contains both  $a$  and  $a'$ ,  $a$  is not  
835 the left-most or right-most point of  $\partial D_h(u)$ . Suppose that  $a$  belongs to the top chain  
836 of  $\partial D_h(u)$ . Then we deduce that  $y(a') < y(a)$ . Since  $D_h(v)$  also contains both  $a$  and  
837  $a'$ , we get that  $a$  belongs to the top chain of  $\partial D_h(v)$ . So  $a$  belongs to the top (resp.,  
838 bottom) chain of  $\partial D_h(u)$  if and only if  $a$  belongs to the top (resp., bottom) chain of  
839  $\partial D_h(v)$ . The same argument applies to  $b$ .

Now there are several possibilities, depending on  $a$  and  $b$  being in the top or bottom  
chains of  $\partial D_h(u)$  and  $\partial D_h(v)$ . The arguments for all cases are similar, so we only  
explain the case where  $a$  and  $b$  belong to the top chains of  $\partial D_h(u)$  and  $\partial D_h(v)$ . In this  
case,  $u$  and  $v$  lie in the portion of  $bis(a, b)$  below the lines  $y = y(a)$  and  $y = y(b)$  (and  
recall that  $y(u) > y(v)$ ) (see Figure 21, center). Either  $g$  or  $g'$  lies in  $D_h(u) \setminus D_h(v)$ ,  
so in particular in the portion of  $D_h(u)$  above the segment  $ab$ . Since  $a$  and  $b$  belong to  
the top chain of  $\partial D_h(u)$ , this portion lies between the lines  $x = x(a)$  and  $x = x(b)$ .  
Thus we obtain  $x(a) < x(g) < x(b)$ .  $\square$



**Fig. 21.** Left: Case where  $x(a) = x(b)$ . Center: Either  $g$  or  $g'$  lies in a portion of  $D_h(u)$  between  
the lines  $x = x(a)$  and  $x = x(b)$ . Right: The segment  $gg'$  is of type middle for  $e_1$ .

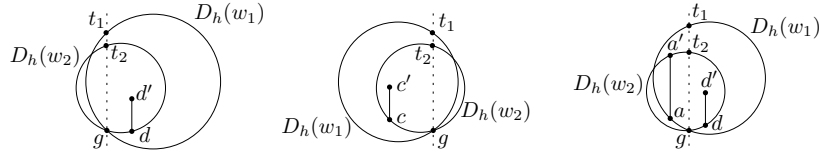
840 We are now ready to prove Lemma 25.

841 *Proof (of Lemma 25).* We assume that all segments in  $S$  are vertical. We proceed by  
842 contradiction. So let us assume that the segment  $gg'$  is of type middle for two pure edges  
843 of  $HVD(S)$ , namely  $e_1$  and  $e_2$ . Let  $e_1$  be in the boundary of  $\text{hreg}(a)$  and  $\text{hreg}(b)$ , for  
844 two segments  $aa', bb' \in S$  (see Figure 21, right). Analogously,  $e_2$  is in the boundary of  
845  $\text{hreg}(c)$  and  $\text{hreg}(d)$ , for two segments  $cc', dd' \in S$ . By Lemma 26a,  $x(a) \neq x(b)$  and  
846  $x(c) \neq x(d)$ . Without loss of generality, we suppose that  $x(a) < x(b)$  and  $x(c) < x(d)$ .  
847 We also assume that  $y(g) < y(g')$ .

848 Consider the disk having  $a, b$  and  $g$  on the boundary; this disk corresponds to a disk  
849  $D_h(w_1)$ , for some point  $w_1$  on the edge  $e_1$ . Analogously, the disk having  $c, d$  and  $g$  on  
850 the boundary corresponds to a disk  $D_h(w_2)$ , for some point  $w_2$  on  $e_2$ .

851 Since, by Lemma 26b,  $x(a) < x(g) < x(b)$ , the line  $x = x(g)$  intersects  $\partial D_h(w_1)$   
852 twice. One of these intersection points is  $g$ , and we next show that the second intersec-

tion point, called  $t_1$ , is above  $g$ . Since the line through  $a$  and  $b$  leaves  $g$  and  $g'$  on opposite sides, and since  $x(a) < x(g) < x(b)$ , the segment  $ab$  intersects the segment  $gg'$ . The intersection point lies above  $g$  and, by convexity, it is contained in  $D_h(w_1)$ . This implies that  $g$  is in the bottom chain of  $\partial D_h(w_1)$  and, consequently,  $t_1$  is above  $g$  (see Figure 22, left). Analogously, the second intersection point  $t_2$  between  $\partial D_h(w_2)$  and  $x = x(g)$  also lies above  $g$ . Without loss of generality, we assume that  $y(t_1) \geq y(t_2)$ . We divide the rest of the argument into several cases.



**Fig. 22.** Left: Case where the four segments are distinct and the second intersection point between  $\partial D_h(w_1)$  and  $\partial D_h(w_2)$  is to the left of  $x = x(g)$ . Middle: Case where the four segments are distinct and the second intersection point between  $\partial D_h(w_1)$  and  $\partial D_h(w_2)$  is to the right of  $x = x(g)$ . Right: Case where  $a = c'$ .

We start by considering the case where the four segments  $aa'$ ,  $bb'$ ,  $cc'$ , and  $dd'$  are distinct. Because  $w_1$  is in the boundary of  $\text{hreg}(a)$  and  $\text{hreg}(b)$  and the four segments are distinct,  $D_h(w_1)$  contains at most one of  $\{c, c'\}$  and at most one of  $\{d, d'\}$ . Analogously,  $D_h(w_2)$  contains at most one of  $\{a, a'\}$  and at most one of  $\{b, b'\}$ . Consequently, none of  $D_h(w_1)$ ,  $D_h(w_2)$  contains the other, and  $\partial D_h(w_1)$  and  $\partial D_h(w_2)$  intersect at  $g$  and at a second point. If this second point lies to the left of the line  $x = x(g)$ , then the portion of  $D_h(w_2)$  to the right of  $x = x(g)$  is contained in  $D_h(w_1)$  (see Figure 22, left). Consequently,  $dd'$  is in  $D_h(w_1)$ , yielding a contradiction. If the second intersection point between  $\partial D_h(w_1)$  and  $\partial D_h(w_2)$  lies to the right of the line  $x = x(g)$ , then  $cc'$  is in  $D_h(w_1)$  (see Figure 22, center). If the intersection point lies on  $x = x(g)$ , then  $y(t_1) = y(t_2)$ , and we obtain that  $D_h(w_1)$  contains  $cc'$  or  $dd'$ .

Let us look at the remaining cases. Since  $x(a) < x(g) < x(b)$  and  $x(c) < x(g) < x(d)$ , the intervals  $(x(a), x(b))$  and  $(x(c), x(d))$  have non-empty intersection. This implies that  $aa' \neq dd'$  and  $bb' \neq cc'$  (and obviously  $aa' \neq bb'$  and  $cc' \neq dd'$ ). Therefore, the two remaining cases are  $aa' = cc'$  and  $bb' = dd'$ . We divide the first one into two subcases, namely,  $a = c$  and  $a = c'$ .

If  $a = c$ , the second intersection point between  $\partial D_h(w_1)$  and  $\partial D_h(w_2)$  is  $a$ , which lies to the left of the line  $x = x(g)$ . If  $bb' \neq dd'$ , then  $dd'$  is contained in  $D_h(w_1)$ , yielding a contradiction. If  $bb' = dd'$ , then, since  $b$  lies on  $\partial D_h(w_1)$  and  $d$  lies on  $\partial D_h(w_2)$ , we have that  $b \neq d$ . But then  $b = d'$  lies on the portion of  $\partial D_h(w_1)$  to the right of  $x = x(g)$ , that is, outside  $D_h(w_2)$ , yielding a contradiction.

If  $a = c'$ , due to the assumption that  $y(t_1) \geq y(t_2)$ , we have that  $a$  is in the bottom chain of  $\partial D_h(w_1)$  and  $a' = c$  is in the top chain of  $\partial D_h(w_2)$  (see Figure 22, right). Then the second intersection point between  $\partial D_h(w_1)$  and  $\partial D_h(w_2)$  lies to the left of

884  $x = x(a)$ , and we also have that  $dd'$  is in  $D_h(w_1)$  (if  $bb' \neq dd'$ ) or that  $d'$  is outside  
 885  $D_h(w_2)$  (if  $bb' = dd'$ ).

In the last case,  $bb' = dd'$ . This case is symmetric to the previous one, and it also yields a contradiction.  $\square$

## 886 6 Lower Bound

887 Below we prove a lower bound for computing a stabbing circle of minimum radius for  
 888 sets of segments which are parallel and have equal length.

889 **Theorem 5.** *The problem of computing a stabbing circle of minimum radius for a set*  
 890 *of  $n$  parallel segments of equal length has an  $\Omega(n \log n)$  lower bound in the algebraic*  
 891 *decision tree model.*

892 *Proof.* The reduction, very similar to that of Theorem 6 in [3], is from  $\text{MAXGAP}(X)$ .  
 893 In our version, the input  $X$  consists of a set of  $n$  integers  $x_1, \dots, x_n$ , and  $\text{MAXGAP}(X)$   
 894 is the problem of finding the maximum difference between consecutive elements of  $X$ .

895 Without loss of generality, we may assume  $\min X = 1$ . Let  $x'_1 < x'_2 < \dots < x'_n$   
 896 be the sorting of the elements of  $X$ . Then  $x'_1 = 1$ , and let  $M = x'_n$ . We construct a set  
 897  $S$  of parallel segments of equal length as follows: For every  $x_i \in X$ , we add a segment  
 898 connecting point  $(x_i, 0)$  to  $-(M + 1) + x_i, 0$ . Additionally, we add two segments  
 899  $aa'$  and  $bb'$  such that  $a = (-1/2, 0)$ ,  $a' = -(M + 1) - 1/2, 0$ ,  $b = (1/2, 0)$ , and  
 900  $b' = ((M + 1) + 1/2, 0)$ .

901 Any stabbing circle for  $S$  of minimum radius contains  $a, b$  in its interior. Thus  
 902 the possibilities for such a stabbing circle are: If the associated disk contains  
 903  $a, b, (x'_1, 0), \dots, (x'_n, 0)$ , or  $-(M + 1) + x'_1, 0, \dots, -(M + 1) + x'_n, 0, a, b$ , then  
 904 it has diameter  $M + 1/2$ . If it contains  $-(M + 1) + x'_{i+1}, 0, \dots, -(M + 1) +$   
 905  $x'_n, 0, a, b, (x'_1, 0), \dots, (x'_i, 0)$  for  $i < n$ , then it has diameter  $M + 1 - (x'_{i+1} - x_i)$ .  
 906 Since  $\text{MAXGAP}(X) \geq 1$ , the stabbing circles of minimum radius belong to the last  
 907 family. Thus  $\text{MAXGAP}(X)$  is equivalent to finding the stabbing circle for  $S$  of mini-  
 908 mum radius.

The set  $S$  does not satisfy all the assumptions of this paper, since all endpoints are collinear. We construct a set  $S'$  obtained from  $S$  by translating every segment vertically by distinct values of at most  $\varepsilon = 1/10$ . Since  $\varepsilon$  is small compared to the difference between distinct values of diameters of different stabbing circles for  $S$  (which is at least  $1/2$ ), a minimum stabbing circle for  $S'$  corresponds to a minimum stabbing circle for  $S$  which is combinatorially “the same”. This proves that the lower bound also holds for the more restricted sets of segments considered in this paper.  $\square$

## 909 7 Conclusions and Future Work

910 The connection between the stabbing circle problem and the cluster Voronoi diagrams  
 911 allows to solve the stabbing circle problem in an efficient way under certain conditions  
 912 on  $S$ . In fact, this connection is a good tool for studying the stabbing circle problem  
 913 for segment sets of particular types. So our goal is “to understand” the time and space

complexities depending on the kind of segment sets we work with. In this paper, we have shown that our method allows to solve the stabbing circle problem in  $o(n^2)$  time (that is, faster than in the general case) when all segments are parallel. The further open question is to investigate other segment sets for which this is also the case.

A very recent short abstract [13] is a preliminary step in this direction. We consider a set  $S$  of  $n$  disjoint segments that correspond to edges of the Delaunay triangulation of some fixed point set, and we show that the stabbing circle problem for  $S$  can be solved in  $O(n \log n)$  time and  $O(n)$  space in two cases: (i) all segments in  $S$  are parallel; (ii) all segments in  $S$  have equal length. It remains as an open problem whether the stabbing circle problem for such a set  $S$  (not necessarily satisfying (i) or (ii)) can be solved in subquadratic time. Another case for which it might be possible to achieve near-linear time is that of a set of pairwise-disjoint segments which are vertical or horizontal, or which are aligned to a given (constant) set of directions.

In a different vein, it would be interesting to find a way to charge the time and space complexities required to solve the stabbing circle problem directly to the complexity of  $\text{FCVD}^*(S)$ , rather than to the number of segments of type middle for  $S$ .

Further, the find-change query in this paper is implemented in a straight forward way. It is an open problem whether this query could be handled more efficiently, in say  $O(\log n)$  time. If this question was answered in the affirmative, the time complexity of our algorithms would improve by a logarithmic factor.

Finally, deriving a lower bound for the decision version of the stabbing circle problem remains open.

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